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ON CERTAIN DETERMINANT FORMS AND THEIR APPLICATIONS.*

(SECOND PAPER.)

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1. In my first paper on this subject (Annals of Mathematics, VI, 5) I presented in an elementary manner certain determinant forms treated by a process based on a generalization of Rolle's theorem, there given, which lead to a new method of treatment of these forms, and which culminated in a general functional relation formula at the close of the paper, which was left there without further comment than simply its brief demonstration. This formula which I have called a *composite* is one of much generality and great power when applied to the expansion of functions in series. It is as an expander of functions in converging infinite series, that I propose to discuss it in the present paper, limiting the immediate investigation to the expansion of explicit functions.

The first section presents the general statement and demonstration of the theorem, with an outline sketch of the method of its application; the second treats of the expansion of an arbitrary function fx in terms of rational integral functions of x, illustrating expansion in terms of that class of functions which yield a body-determinant unity; while the third section considers the expansion of the function in terms of certain transcendental functions, especially exponential forms and trigonometrical functions, illustrating expansion in terms of that class of functions which give the body-determinant as a difference-product.

I. THE COMPOSITE AND ITS DEMONSTRATION.

2. We have

$$\begin{vmatrix} f_1x, & f_2x, & f_3x, & \ldots, & f_nx, & f_{n+1}x \\ f_1y_1, & f_2y_1, & f_3y_1, & \ldots, & f_ny_1, & f_{n+1}y_1 \\ f_1y_2, & f_2y_2, & f_3y_2, & \ldots, & f_ny_2, & f_{n+1}y_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_1y_p, & f_2y_p, & f_3y_p, & \ldots, & f_ny_p, & f_{n+1}y_p \\ f_1'x_1, & f_2'x_1, & f_3'x_1, & \ldots, & f'_nx_1, & f'_{n+1}x_1 \\ f_1''x_2, & f_2''x_2, & f_3''x_2, & \ldots, & f''_nx_2, & f''_{n+1}x_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_1^qx_q, & f_2^qx_q, & f_3^qx_q, & \ldots, & f_n^qx_q, & f_{n+1}^qx_q \\ \Phi(u), & 0, & 0, & \ldots, & 0, & 1 \end{vmatrix} \equiv 0, \qquad (11)$$

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*Read before the Mathematical Section of the University of Virginia Philosophical Society.

wherein $p + q = n - 1$, the functions $y_1, \ldots, y_p, x_1, \ldots, x_q$ are independent of x; the notation $f_s^r x_t$ means that $f_s x$ is to be differentiated r times with respect to x; and in the result, x is to be replaced by x_t.

Also

$$\Phi(u) = (-1)^n \frac{D_u^{q+1}}{M_u^{q+1}},$$

wherein D is the determinant derived from the above determinant (11) by the deletion of its last row and column, and the notation D_u^{q+1} means that the $(q+1)$th derivative of D with respect to x is to be taken and in the result x replaced by u. The determinant M is that one obtained by the deletion of the first column and last row in (11), and with a corresponding meaning attached to M_u^{q+1}.

The quantity u is some unknown quantity which lies between the greatest and the least of the quantities $x, x_1, \ldots, x_q, y_1, \ldots, y_p$.

The usual assumptions are made with regard to the continuity and finiteness of the functions dealt with.

The determinant (11) I ventured to name a composite, inasmuch as it is composed partially of the form of an alternant, and partially of a form which is that of a Wronskian. With this distinction, however, in the case of the latter; that the quantities under the derived functional signs may in general be entirely independent of each other, and are independent of the corresponding quantity x under the primitive functional sign.

The proof of the theorem is very simple, and as follows:—

In the determinant (11), strike out the last row and column, calling the result D.

Let

$$D = MR,$$

where M is the determinant left after striking out the first column and the last row of (11), and R is some unknown function of x.

Let x_0 be some fixed value of x. Substitute x_0 for x in $D = MR$, and we have $D_0 = M_0 R_0$ (using the subscript to denote that x has been changed into x_0).

Consider the function

$$D - MR_0,$$

which for brevity we shall call φ. Indicating the derivatives of φ with respect to x by $\varphi', \varphi'' \ldots, \varphi^n$, we have $\varphi = 0$ for $x = y_1, \ldots, y_p$, also for $x = x_0$. Hence the first derivative of φ must vanish for some value of x between the greatest and the least of the quantities $x_0, y_1, \ldots, y_p$ (by Rolle's theorem), say u_1.

But, φ' vanishes for $x = x_1$, therefore φ'' must vanish for some value of x, say u_2, which lies between u_1 and x_1. And so on, until finally the $(q + 1)$th derivative of φ must vanish for some value of x, say u, such that u lies between u_q and x_q, or in fact, between the greatest and the least of the quantities $x_0, y_1, \ldots, y_p, x_1, \ldots, x_q$. Whence

$$D_u^{q+1} - M_u^{q+1} R_0 = 0 .$$

Therefore

$$D_0 = M_0 \frac{D_u^{q+1}}{M_u^{q+1}} ,$$

or

$$D_0 + (-1)^n \Phi(u) M_0 = 0 ,$$

letting $(-1)^n \Phi(u)$ represent the ratio D_u^{q+1}/M_u^{q+1}. Now if x_0 be *any* value of x for which fx and its $q + 1$ derivatives are finite and continuous, then will this last relation be true for all such values of x. We may therefore drop the suffix, and write generally

$$D + (-1)^n M \Phi(u) = 0 ,$$

which establishes the proof* of the formula (11).

In order to exhibit more clearly the nature of the preceding general formula, consider it in the following manner. Let us assume that the possibility of the development of a function fx in terms of the functions $\varphi_1 x$, $\varphi_2 x$, etc., has been demonstrated, so that we have the development

$$fx = a_0 + a_1 \varphi_1 x + a_2 \varphi_2 x + \ldots + a_n \varphi_n x + \ldots ,$$

in which the coefficients a are constants to be determined.

Consider the function Fx defined by the first n terms of the above series, so that we have

$$Fx \equiv a_0 + a_1 \varphi_1 x + a_2 \varphi_2 x + \ldots + a_n \varphi_n x .$$

Suppose that the functions $\varphi_1 x, \ldots, \varphi_n x$ are such that the operation of differentiation* may be performed upon them n times. Then, differentiating this last identity n times, we form the n equations

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* Everyone will recognize in this demonstration the method of Homersham Cox's proof of Taylor's formula as given in Todhunter's Calculus, the Encyclopædia Britannica, and elsewhere.

$$F'x = a_1\,\varphi_1'x + a_2\,\varphi_2'x + \ldots + a_n\,\varphi_n'x\,,$$

$$F''x = a_1\,\varphi_1''x + a_2\,\varphi_2''x + \ldots + a_n\,\varphi_n''x\,,$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$F^n x = a_1\,\varphi_1^n x + a_2\,\varphi_2^n x + \ldots + a_n\,\varphi_n^n x\,.$$

Solving these for the a's, we have

$$a_r = \frac{|\,\varphi_1'x,\ \varphi_2''x,\ \ldots,\ F^r x,\ \ldots,\ \varphi_n^n x\,|}{|\,\varphi_1'x,\ \ldots,\ \varphi_n^n x\,|}\,,$$

$$= (-1)^{r+1}\sum_{p=1}^{p=n} F^p x\,\frac{\varDelta(r,p)}{\varDelta}\text{†}.$$

Since no restriction has been placed on the value of n, we may make n as large as we choose. When n approaches infinity, Fx converges to fx as a limit, and the value of a_r is

$$a_r = (-1)^{r+1}\sum_{p=1}^{p=\infty} f^p x\,\frac{\varDelta(r,p)}{\varDelta}\,.$$

Since in each of the equations above, we may substitute for x after differentiation any arbitrary constant (which does not disturb the equality sign), we have for the value of a_r the same expression as will be given by the expansion of the composite. The foregoing illustration is very instructive ; it is the method employed by Fourier in discovering his celebrated expansion in terms of sines (Théorie de la Chaleur). We may regard this method of procedure as the equivalent of the ordinary method of indeterminate coefficients when applied to the expansion of functions in series, which when the series has been shown to be possible, furnishes us a means of determining the values which the constant coefficients must have. As to the possibility of such a development we proceed now to further examine.

3. The particular case of the composite which we propose to discuss here,

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* We are not limited here to the operation of differentiation, for we may, instead of differentiation, perform any other operation admissible on these functions, and get the same results. We may thus perform the operation of finite differences, indicated by the symbol $\varDelta$; or that of enlargement, indicated by E. Indeed any operation, say indicated by §, leads to the same result ; but that result may be absurd, that is to say, a mere analytical form without arithmetical meaning.

† For the notation $\varDelta(r,p)$ and $\varDelta$, see below.

is that in which $p = 1$, and f_2x, f_2y_1 are also each unity, so that we have for our present purpose,

$$\begin{vmatrix} fx & , 1, & \varphi_1 x & , \varphi_2 x & , \ldots, & \varphi_n x & , \varphi_{n+1} x \\ fy & , 1, & \varphi_1 y & , \varphi_2 y & , \ldots, & \varphi_n y & , \varphi_{n+1} y \\ f' x_1 & , 0, & \varphi'_1 x_1 & , \varphi_2' x_1 & , \ldots, & \varphi_n' x_1 & , \varphi_{n+1}' x_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f^n x_n & , 0, & \varphi_1^n x_n & , \varphi_2^n x_n & , \ldots, & \varphi_n^n x_n & , \varphi_{n+1}^n x_n \\ \Phi(u) & , 0, & 0 & , 0 & , \ldots, & 0 & , 1 \end{vmatrix} \equiv 0 . \qquad (12)$$

The general outline of treatment is as follows:—

Expanding (12) with respect to the last row we get two determinants; the term involving $\Phi(u)$ we always represent by R. The second of these determinants we break up into two others by means of its second column. One of these is a function of y and is independent of x, we therefore regard it as a constant term, and represent it by Fy when it becomes necessary to refer to it briefly. The second of these is a function of x which we represent by Fx, noticing that x enters it identically as y enters Fy, we shall therefore generally find it necessary to discuss only Fx in determining the form of the expansion.

Writing out, we may have the form briefly expressed as

$$Fx \equiv Fy + R'. \qquad (13)$$

The function Fx is the one which will chiefly demand our attention. Regarding this, we observe that it may be written out in either of two ways: first, with respect to its first row, in which case we have a series of terms arranged according to the φx functions; second, we may write it out with regard to the first column, and thus obtain a series of terms arranged according to the derived functional forms $f'x_1, f''x_2, \ldots$. We shall usually make all of the arbitrary constants equal to some single arbitrary z.

In general, to expand an arbitrary function fx in terms of functions $\varphi_1 x$, $\varphi_2 x, \ldots$, etc., consider Fx in (13). This determinant may be written out with respect to either first row or column. Represent by Δ the determinant Fx in which its first row and column have been struck out. Represent by $\Delta(r, p)$ the determinant Δ in which its rth column and pth row have been struck out. We then have for the expansion of Fx with respect to its first row,

$$\frac{Fx}{\Delta} \equiv fx - \sum_{r=1}^{r=n} (-1)^{r+1} \varphi_r x \sum_{p=1}^{p=n} (-1)^{p+1} \frac{\Delta(r, p)}{\Delta} f^p z, \qquad (14)$$

and with respect to its first column

$$\frac{Fx}{\Delta} = fx - \sum_{p=1}^{p=n} (-1)^{p+1} f^p z \sum_{r=1}^{r=n} (-1)^{r+1} \frac{\Delta(r,p)}{\Delta} \varphi_r x\,. \tag{15}$$

By the aid of which we write down at once the expansion of (12) with respect to first row and column respectively as below,

$$fx - fy = \sum_{r=1}^{r=n} (-1)^{r+1} (\varphi_r x - \varphi_r y) \sum_{p=1}^{p=n} (-1)^{p+1} \frac{\Delta(r,p)}{\Delta} f^p z + R\,, \tag{16}$$

$$fx - fy = \sum_{p=1}^{p=n} (-1)^{p+1} f^p z \sum_{r=1}^{r=n} (-1)^{r+1} \frac{\Delta(r,p)}{\Delta} (\varphi_r x - \varphi_r y) + R\,. \tag{17}$$

In order that we may write this formula to infinity (that it shall become a convergent infinite series), R must be evanescent when n is infinite, and the successive terms begin with finite values and diminish in absolute magnitude after some fixed term, according to the laws which define convergent series.

In general the first point for consideration is the ratio $\Delta(r,p)/\Delta$, which must be finite when n is infinite, for small values of r and p; and become evanescent, when n is infinite, for infinitely large values of r and p. This being so, then the coefficients of $\varphi_r x$ or $f^p z$ in the series (14) or (15), respectively, will be convergent series with arithmetical values whenever the functions we are dealing with are finite and continuous for the values of the argument indicated. Yet some of the series which we shall derive waive this injunction of continuity.

It will soon be seen that the application of the composite to the expansion of functions after the manner above, is limited only by our present imperfect knowledge of determinants when we come to evaluate the ratio $[\Delta(r,p)/\Delta]_{n=\infty}$.

Although the ratio $[\Delta(r,p)/\Delta]_{n=\infty}$ may be determined and found to be such as to satisfactorily fulfil the law for converging series; yet, at no time, have we the right to leave off the R and extend the summation to infinity without having previously ascertained that R becomes evanescent when n becomes infinity, thus showing that the quantity under the summation sign converges to the member on the left as a limit as n becomes infinitely great. Under these circumstances only can the member on the right, when written to infinity, without the R, be regarded as the arithmetical equivalent of the member on the left of the equality sign in series (16) or (17).

The examination of the remainder for the general series is not a simple matter. In the case of those series of the first class the remainder can at once be written down and examined; for the higher order of series the general form of the remainder may, of course, be given; but until the form of the series-functions is given, we cannot tell what becomes of R when n is infinite.

Using the same nomenclature as in § 2, we may get an expression for R, as follows: The determinant got by striking out the first column and row of the composite we call the *body-determinant* of the composite, and will always represent it by the symbol Δ.

Disregarding the sign of the remainder, we have from (12)

$$R' = M\ \Phi(u).$$

Since Δ is the coefficient of fy and fx in the expansion, and as we shall always divide through by Δ, we may put for the remainder

$$R = \frac{M}{\Delta}\Phi(u),$$

$$= \frac{M}{\Delta}\,\frac{D_u^{n+1}}{M_u^{n+1}}.$$

Since $D = Fx - Fy$, we have $D_u^{n+1} = F^{n+1}(u)$; whence

$$R = \frac{M}{M_u^{n+1}}\,\frac{F^{n+1}(u)}{\Delta},$$

$$= \frac{M}{M_u^{n+1}}\left[f^{n+1}(u) - \overset{r=n}{\underset{r=1}{\Sigma'}}(-1)^{n+1}\varphi_r^{n+1}(u)\overset{p=n}{\underset{p=1}{\Sigma'}}(-1)^{p+1}\frac{\Delta(r,p)}{\Delta}f^{p}z\right]. \qquad \text{(i)}$$

We derive another and more convenient form of the remainder by expanding (12) by its second column, then each term by its first row; whence

$$R' = \Delta\,(\varphi_{n+1}x - \varphi_{n+1}y)\ \Phi(u);$$

and on dividing the series through by Δ, we obtain

$$R = (\varphi_{n+1}x - \varphi_{n+1}y)\ \Phi(u) \qquad \text{(ii)}$$

$$= (\varphi_{n+1}x - \varphi_{n+1}y)\frac{\Delta}{M_u^{n+1}}\left[f^{n+1}(u) - \overset{r=n}{\underset{r=1}{\Sigma'}}(-1)^{n+1}\varphi_r^{n+1}(u)\overset{p=n}{\underset{p=1}{\Sigma'}}(-1)^{p+1}\frac{\Delta(r,p)}{\Delta}f^{p}z\right].$$

In this case we have to notice that the coefficients of the φ functions themselves are infinite series, which have remainders involving $\Phi(u)$, and which require examination in like manner.

We notice that the expression within the brackets in (i) and (ii) is but the $(n+1)$th derivative of the first n terms of the series, u replacing x after differentiation; if, therefore, the series is a convergent one, then when n is infinite this expression vanishes, and R is a zero identity in virtue of this relation. The limiting value of R, however, must be obtained through investigation.

In the following two sections I shall make no attempt to investigate the *remainder* or the term represented by R. I have thought it best to devote them to the deducing of special forms of series with the immediate purpose in view of illustrating a general method of operation in deducing the special forms of series, reserving the direct investigation of R for a separate paper.

In order to be explicit, we repeat that R is the symbol used to represent the terminal term when the determinant (12) is expanded with respect to either first row or first column, in the following forms respectively:

$$fx = fy + A_1\varphi_1 x + \ldots + A_n\varphi_n x + R\,,$$

$$fx = fy + B_1 f'x_1 + \ldots + B_n f^n x_n + R\,.$$

The symbol R, being merely a general symbol meaning the remainder of the series after the terms $A_n\varphi_n x$ or $B_n f^n x_n$, is not to be taken as having the same value in the two cases. The coefficients A are independent of the variable x, while the coefficients B do not involve the *form* of the function fx.

4. In applying the composite to the expansion of functions we shall have use for the following general theorem:

If in (11) the functions $f_1 x$, $f_2 x$, etc. be such that they may each be expressed by an infinite converging series in terms of integral powers of the variable, and if the rth power of the variable be not included in any of these series, then the row of the composite which is composed of the rth derivative forms of these functions must be omitted from the composite.*

In order to prove this we take up Taylor's formula (4), thus,

$$f(x+h) = fh + xf'h + \tfrac{1}{2!}x^2 f''h + \ldots$$

If the rth power of the variable be not contained in the series, then

$$f^r(x+h) = xf^{r+1}h + \tfrac{1}{2!}x^2 f^{r+2}h + \ldots,$$

in which, if $x = 0$, we have

$$f^r h = 0\,.$$

Hence, in the composite, if after differentiation we put $x = 0$, then all of the constituents in the row of rth derivatives vanish, and the composite vanishes identically. This row must therefore be omitted from the composite.

That the truth of the general theorem still holds when this row is omitted, we may see by going back to the demonstration given in § 2. The analysis is the same, word for word, until we reach the row in question, when we find that the $(r-1)$th derivative of the function φ vanishes for some value of x, say u_{r-1}, which lies between x_{r-2} and u_{r-2}, and also vanishes for $x = x_{r-1}$. Hence the rth derivative of φ must vanish for $x = u_r$, where u_r lies between u_{r-1} and x_{r-1}. But the rth derivative of φ vanishes when $x = 0$, because all of the first row becomes zero when $x = 0$, by the above. Hence the $(r+1)$th derivative of φ vanishes for u_{r+1} which lies between 0 and u_r, and so on. The only change in the result being that we must include zero among the arbitraries between the greatest and least of which u lies.

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* The same theorem, of course, applies to any set of functions which are such that the rth derivative row vanishes when a particular value be substituted for the variable after differentiation. An important application of this principle is given later, wherein this particular value of the variable is infinity.

The same method of reasoning applies to any number of missing powers of the variable. If the functions contain only the even powers of the variable, the composite cannot contain the odd derivative forms. If they contain only the odd powers, the composite contains no even derivative forms. This important theorem will be useful in expanding such forms.

II. Rational Integer-Power Series.

5. The forms included in the first class of functions to which we give attention comprise all of those functions which are such that their derivatives on one side of the main diagonal of the composite can be made to vanish. In this case the body-determinant is always unity, and the coefficients in the expansion are determinants which may be written out into recurrence formulæ. The rational integral function is the type of this class. Examples of other functions than rational integers which belong to this class will also be given.

Since we may factor out the coefficient of the highest power of x, we may, without loss of generality, use the rational integral function in the general shape

$$\varphi_n x = x^n + p_1 x^{n-1} + \ldots + p_n ,$$

and consider the expansion of an arbitrary function fx in terms of the rational integral functions $\varphi_r x/r!$, where r takes successively the values $1, 2, 3, \ldots.$

We have, then, the following general theorem:

$$\begin{vmatrix} fx, & 1, & \varphi_1 x, & \frac{\varphi_2 x}{2!}, & \frac{\varphi_3 x}{3!}, & \ldots, & \frac{\varphi_n x}{n!}, & \frac{\varphi_{n+1} x}{(n+1)!} \\ fy, & 1, & \varphi_1 y, & \frac{\varphi_2 y}{2!}, & \frac{\varphi_3 y}{3!}, & \ldots, & \frac{\varphi_n y}{n!}, & \frac{\varphi_{n+1} y}{(n+1)!} \\ f'x_1, & 0, & 1, & \varphi_2' x_1, & \frac{\varphi_3' x_1}{2!}, & \ldots, & \frac{\varphi_n' x_1}{(n-1)!}, & \frac{\varphi_{n+1}' x_1}{n!} \\ f''x_2, & 0, & 0, & 1, & \varphi_3'' x_2, & \ldots, & \frac{\varphi_n'' x_2}{(n-2)!}, & \frac{\varphi_{n+1}'' x_2}{(n-1)!} \\ f'''x_3, & 0, & 0, & 0, & 1, & \ldots, & \frac{\varphi_n''' x_3}{(n-3)!}, & \frac{\varphi_{n+1}''' x_3}{(n-2)!} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f^n x_n, & 0, & 0, & 0, & 0, & \ldots, & 1, & \varphi_{n+1}^{n} x_n \\ \Phi(u), & 0, & 0, & 0, & 0, & \ldots, & 0, & 1 \end{vmatrix} = 0. \qquad (18)$$

This when expanded with respect to the first column becomes

$$fx = fy - C_1 f'x_1 + C_2 f''x_2 - C_3 f'''x_3 + \ldots + (-1)^n C_n f^n x_n + R, \qquad (19)$$

wherein the coefficients C_1, C_2, ..., are to be determined from the recurrence formula

$$C_r = C_{r-1}\,\varphi_r^{r-1} x_{r-1} - \tfrac{1}{2!}\,C_{r-2}\,\varphi_r^{r-2} x_{r-2} + \tfrac{1}{3!}\,C_{r-3}\varphi_r^{r-3} x_{r-3} - \cdots$$
$$+ (-1)^{r+1}\,\frac{\varphi_r x - \varphi_r y}{r!}. \qquad (20)$$

Now $\varphi_r x$ contains r arbitrary constants, and at each differentiation an additional constant is introduced; so that $\varphi_r x$ and its derivatives contain $2r-1$ arbitraries; and if in this list we include the y of $\varphi_r y$, then the $(r+2)$th column contains $2r$ arbitraries. Hence it follows that the coefficient C_r contains $(r-1)(r+2)$ arbitrary constants, of which $r-2$ are held in common with C_{r-1}. So that each term of the series brings in r^2 new arbitrary constants, all of which are at our disposal for the creation of special forms. If we choose we may write the constants so introduced by each term as a complete determinant.

If we expand (18) according to the form (13), then considering Fx, we may write this out with respect to either first row or column.

Thus writing with respect to the first column, we get,

$$fx = fy + B_1 f'x_1 + B_2 f''x_2 + B_3 f'''x_3 + \ldots + R. \qquad (21)$$

In which $B_r = A_r(x) - A_r(y)$, and

$$A_r(x) = (-1)^{r+1} \begin{vmatrix} \varphi_1 x, & \dfrac{\varphi_2 x}{2!}, & \dots, & & \dfrac{\varphi_r x}{r!} \\ 1, & \varphi_2' x_1, & \dots, & & \dfrac{\varphi_r' x_1}{(r-1)!} \\ 0, & 1, & \dots, & & \dfrac{\varphi_r'' x_2}{(r-2)!} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & \dots, & 1, & \varphi_r^{r-1} x_r \end{vmatrix}.$$

Replacing x by y in this we have $A(y)$, which may be written as a recurrence formula. We now pass to the consideration of simpler forms.

6. Consider the expansion of fx in terms of the functions $x^r/r!$; then we have

$$\begin{vmatrix} fx, & 1, & x, & \dfrac{x^2}{2!}, & \dots, & \dfrac{x_n}{n!}, & \dfrac{x^{n+1}}{(n+1)!} \\ fa_0, & 1, & a_0, & \dfrac{a_0^{\,2}}{2!}, & \dots, & \dfrac{a_0^{\,n}}{n!}, & \dfrac{a_0^{\,n+1}}{(n+1)!} \\ f'a_1, & 0, & 1, & a_1, & \dots, & \dfrac{a_1^{\,n-1}}{(n-1)!}, & \dfrac{a_1^{\,n}}{n!} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f^n a_n, & 0, & 0, & 0, & \dots, & 1, & a_n \\ f^{n+1}(u), & 0, & 0, & 0, & \dots, & 0, & 1 \end{vmatrix} = 0, \qquad (22)$$

a formula which was given with an independent demonstration in my first paper, and which we now propose to study somewhat in detail. In this formula, to repeat, u is some unknown quantity which lies between the greatest and the least of the quantities, $a_0, a_1, \dots, a_n, x$, and $f^r a_r$ means that fx is to be differentiated r times with respect to x, and in the result x changed into a_r.

This determinant may be written out thus:

$$\begin{aligned} fx = fa_0 - A_1 f'a_1 + A_2 f''a_2 - \dots \\ + (-1)^n A_n f^n a_n + (-1)^{n+1} A_{n+1} f^{n+1}(u), \qquad (23) \end{aligned}$$

wherein the coefficient A_r is the determinant which is obtained from the above determinant by the deletion of the first column and the $(r+2)$th row. Thus,

$$A_r = \begin{vmatrix} 1, & x, & \frac{x^2}{2!}, & \ldots, & \frac{x^r}{r!} \\ 1, & a_0, & \frac{a_0^2}{2!}, & \ldots, & \frac{a_0^r}{r!} \\ 0, & 1, & a_1, & \ldots, & \frac{a_1^{r-1}}{(r-1)!} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & 0, & \ldots, 1, & a_{r-1} \end{vmatrix} \tag{24}$$

This may be written out as a recurrence formula ; thus,

$$A_r = a_{r-1} A_{r-1} - \frac{a_{r-2}^2 A_{r-2}}{2!} + \frac{a_{r-3}^3 A_{r-3}}{3!} - \ldots$$
$$+ (-1)^{r+1} \frac{a_1^{r-1}}{(r-1)!} + (-1)^r \frac{x^r - a^r}{r!}. \tag{25}$$

This function A_r is a rational integral homogeneous function of the $r+1$ quantities $x, a_0, \ldots, a_{r-1}$. Its degree is r, and it contains 2^r terms. For convenience of reference the first four values are written down,

$$A_1 = -(x - a_0),$$
$$A_2 = +\tfrac{1}{2!}(x^2 - a_0^2) - a_1(x - a_0),$$
$$A_3 = -\tfrac{1}{3!}(x^3 - a_0^3) + \tfrac{1}{2!}a_2(x^2 - a_0^2) - a_1 a_2(x - a_0) + \tfrac{1}{2!}a_1^2(x - a_0),$$
$$A_4 = +\tfrac{1}{4!}(x^4 - a_0^4) - \tfrac{1}{3!}a_3(x^3 - a_0^3) + \tfrac{1}{2!}a_2 a_3(x - a_0^2) - a_1 a_2 a_3(x - a_0)$$
$$+ \tfrac{1}{2!}a_1^2 a_3(x - a_0) + \tfrac{1}{4}a_2^2(x^2 - a_0^2) - \tfrac{1}{2!}a_1 a_2^2(x - a_0) - \tfrac{1}{3!}a_1^3(x - a_0).$$

In actually computing the values of A_r it will generally be best to use formula (25). We observe that A_r vanishes when $x = a_0$; we may therefore factor out $x - a_0$ from A_r under all circumstances.

7. A_r may be written

$$\begin{vmatrix} a_0, & \frac{a_0^2}{2!}, & \ldots, & \frac{a_0^r}{r!} \\ 1, & a_1, & \ldots, & \frac{a_1^{r-1}}{(r-1)!} \\ 0, & 1, & \ldots, & \frac{a_2^{r-2}}{(r-2)!} \\ \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & \ldots, 1, & a_{r-1} \end{vmatrix} - \begin{vmatrix} x, & \frac{x^2}{2}, & \ldots, & \frac{x^r}{r!} \\ 1, & a_1, & \ldots, & \frac{a_1^{r-1}}{(r-1)!} \\ 0, & 1, & \ldots, & \frac{a_2^{r-2}}{(r-2)!} \\ \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & \ldots, 1, & a_{r-1} \end{vmatrix}. \tag{26}$$

Differentiating with respect to x we have for the derivative of A_r.

$$-\begin{vmatrix} 1, & x, & \frac{x^2}{2!}, & \dots, & \frac{x^{r-1}}{(r-1)!} \\ 1, & a_1, & \frac{a_1^2}{2!}, & \dots, & \frac{a_1^{r-1}}{(r-1)!} \\ 0, & 1, & a_2, & \dots, & \frac{a_2^{r-2}}{(r-2)!} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & 0, & \dots, 1, & a_{r-1} \end{vmatrix} . \tag{27}$$

So that in the process of differentiation the sign of the determinant has been changed; the quantity a_0 eliminated, and the highest power of x deleted. The form of the function is left unaltered. Evidently, then, in order to obtain A_{r-1}, we need only differentiate A_r, and at the same time diminish all suffixes in the result by unity and change the sign.

Indicating this whole operation, excepting the change of sign, by $\mathfrak{d}$, we have

$$A_{r-1} = -\frac{\mathfrak{d}}{dx} A_r, \text{ or simply } A_{r-1} = -\mathfrak{d} A_r .$$

Indicating by a superior prefix the repetition of this operation, we have

$$(-1)^r A_1 = {}^r\mathfrak{d} A_r .$$

We do not usually wish to pass from A's of higher to those of lower order; on the contrary, we wish the reverse process. To pass from a lower A to the next higher order we need only perform the inverse operation to $\mathfrak{d}$; i. e. integrate with respect to x and introduce the arbitrary a_0. Evidently the operation is of the nature of definite integration. If, therefore, we integrate A_{r-1} with respect to x and simultaneously increase all suffixes by unity, the result taken between the limits a_0 and x will be A_r. Thus, if we use I to mean ${}^{-1}\mathfrak{d}$, we have in symbols,

$$A_r = \mathrm{I}_x^{a_0} dx\, A_{r-1} = -\mathrm{I}_{a_0}^{x} dx\, A_{r-1} .$$

Indicating as before the repetition of the operation by a superior prefix, we may write,

$$A_r = (-1)^r \mathrm{I}_{a_0}^{x} dx\, \mathrm{I}_{a_0}^{x} dx \dots \mathrm{I}_{a_0}^{x} dx = (-1)^r\, {}^r\mathrm{I}_{a_0}^{x} dx ,$$

since $A_1 = -(x - a_0)$.

After this manner the coefficients in (23) may be rapidly written down. In our new notation series (23) appears as

$$fx = fa_0 + f'a_1\,{}^1\mathrm{I}^x_{a_0}dx + f''a_2\,{}^2\mathrm{I}^x_{a_0}dx + \dots + f^n a_n\,{}^n\mathrm{I}^x_{a_0}dx + f^{n+1}(u)\,{}^{n+1}\mathrm{I}^x_{a_0}dx. \quad (28)$$

There is no reason why the symbol I should not include the operation on the function fa and its argument, so that

$$\mathrm{I}\,f^r a_r = f^{r+1}a_{r+1}\,;$$

and if we write for symmetry $f^0a = fa$, and agree that

$${}^0\mathrm{I}\,dx\,f^0a = fa\,,$$

we have,

$${}^r\mathrm{I}\,dx\,f^0a_0 = f^r a_r\,.$$

The series may now be written

$$fx = {}^0\mathrm{I}^x_{a_0}dx\,f^0a_0 + {}^1\mathrm{I}^x_{a_0}dx\,f^0a_0 + \dots + {}^n\mathrm{I}^x_{a_0}dx\,f^0a_0 + f^{n+1}(u)\,{}^{n+1}\mathrm{I}^x_{a_0}dx\,;$$

$$= \sum_0^n {}^i\mathrm{I}^x_{a_0}dx\,f^0a_0 + f^{n+1}(u)\,{}^{n+1}\mathrm{I}^x_{a_0}dx\,. \quad (29)$$

Finally, if for brevity we omit to write the remainder after the $(n+1)$th term (assuming the convergency, that the series may be written to infinity), and let x be $a_0 + h$, we have

$$f(a_0 + h) = \Sigma\mathrm{I}\,dx\,fa_0\,.$$

Regarding $\Sigma\mathrm{I}\;dx$ as the operator which turns fa_0 into $f(a_0 + h)$. we have

$$\Sigma\mathrm{I}\,dx = E\,,$$

the fundamental operation of the Calculus of Enlargement.* Omitting the indication of the variable by dx we say briefly that $\Sigma\mathrm{I} = E$.

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* The operation indicated by I, used in the sense in which it is above,—including the operations of integration, differentiation, and change of the arbitrary constant,—admits of a direct interpretation in the particular case

$$\mathrm{I}^a_b\,dx\,fc = (a-b)\,f'(d)\,.$$

This is easily intelligible when the conditions impose d to be that quantity u between a and b such that

$$(a-b)\,f(u) = fa - fb\,,$$

for the operation is then identical with the Δ of the Calculus of Finite Differences. In this sense $\Sigma\mathrm{I} = S = E$.

It may be agreed that I does not act on a pure number, whether it be indicated by symbol or not.

8. We have observed that the rational integral function A_r contains the factor $(x - a_0)$. A little further consideration will show that, if we remove the factor $x - a_0$ from the function A_r, and in the result make $x = a_0$, we obtain

$$\begin{vmatrix} 1, & x, & \frac{x^2}{2!}, & \dots, & \frac{x^{r-1}}{(r-1)!} \\ 1, & a_1, & \frac{a_1^2}{2!}, & \dots, & \frac{a_1^{r-1}}{(r-1)!} \\ 0, & 1, & a_2, & \dots, & \frac{a_2^{r-2}}{(r-2)!} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & 0, & \dots, 1, & a_{r-1} \end{vmatrix}, \tag{30}$$

which is the derivative of A_r with its sign changed. This function A_r' contains $(x - a_1)$ as a factor, which if removed and in the result we place $x = a_1$, we have A_r'', the second derivative of A_r, and so on. In equation (23), if we divide through by $x - a_0$ and denote by B_r the function A_r deprived of its factor $x - a_0$, we obtain

$$\frac{fx - fa_0}{x - a_0} = -B_1 f'a_1 + B_2 f''a_2 - \dots \\ + (-1)^n B_n f^n a_n + (-1)^{n+1} B_{n+1} f^{n+1}(u) \tag{31}$$

In this expression we have to notice that u is in general a function of the quantities $x, a_0, \dots, a_n$, and also involves the form of the function f. Whatever be the nature of the change which goes on in u as the variable x changes its value, the form $f^{n+1}(u)$ in which it enters the above relation will remain unchanged. Therefore, if x converges to a_0, we have, on passing to the limit,

$$f'a_0 = -A_1' f'a_1 + A'_2 f''a_2 - \dots + (-1)^n A'_n f^n a_n + (-1)^{n+1} A'_{n+1} f^{n+1}(u), \tag{32}$$

in which u has not the same value which it had in (31), but now has some value between the greatest and least of the quantities $a_0, \dots, a_n$.

Since a_0 is arbitrary, we may write x for a_0 in (32). Doing this we see that (32) is nothing but the derivative of (23) with respect to x, in which the quantity u behaves as if it were independent of x during the process of differentiation, which therefore shows that (22) or (23) may be differentiated with respect to x, regarding u as a constant in the operation. The result may be treated in like manner to get a second derivative, and so on.

9. Indicate the original function (22) by F, interchange the first and second columns, and write it down thus:

$$F = \begin{vmatrix} fx & , x, & \frac{x^2}{2!}, & \ldots, & \frac{x^{n+1}}{(n+1)!} \\ f'a_1 & , 1, & a_1, & \ldots, & \frac{a_1^n}{n!} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^n a_n & , 0, & 0, & \ldots, 1, & a_n \\ f^{n+1}(u), & 0, & 0, & \ldots, 0, & 1 \end{vmatrix} - \begin{vmatrix} fa_0 & , a_0, & \frac{a_0^2}{2!}, & \ldots, & \frac{a_0^{n+1}}{(n+1)!} \\ f'a_1 & , 1, & a_1, & \ldots, & \frac{a_1^n}{n!} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^n a_n & , 0, & 0, & \ldots, 1, & a_n \\ f^{n+1}(u), & 0, & 0, & \ldots, 0, & 1 \end{vmatrix} = 0.$$

Differentiating, regarding u as a constant, we get

$$\frac{d}{dx} F = \begin{vmatrix} f'x & , 1, & x, & \ldots, & \frac{1}{n!} x^n \\ f'a_1 & , 1, & a_1, & \ldots, & \frac{1}{n!} a_1^n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^n a_n & , 0, & 0, & \ldots, 1, & a^n \\ f^{n+1}(u), & 0, & 0, & \ldots, 0, & 1 \end{vmatrix} = 0 .$$

So that the operation of differentiation eliminates from F the arbitrary constant a_0, a second differentiation eliminates in the same manner a_1, and so on.

Whence

$$\frac{d^n}{dx^n} F = \begin{vmatrix} f^n x & , 1, & x \\ f^n a_n & , 1, & a^n \\ f^{n+1}(u), & 0, & 1 \end{vmatrix} .$$

The inverse operation is, therefore, that of definite integration. Neither operation, however, changes the form of the function.

We have

$$F = \int_{a_0}^{x} \begin{vmatrix} f'x & , 1, & x, & \ldots, & \frac{1}{n!} x^n \\ f'a_1 & , 1, & a_1, & \ldots, & \frac{1}{n!} a_1^n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^n a_n & , 0, & 0, & \ldots, 1, & a_n \\ f^{n+1}(u), & 0, & 0, & \ldots, 0, & 1 \end{vmatrix} dx = 0 .$$

Therefore we may write (22) or (23) thus:

$$\int_{a_0}^{x} dx \int_{a_1}^{x} dx \ldots \int_{a_{n-1}}^{x} dx \begin{vmatrix} f^n x & , 1, & x \\ f^n a_n & , 1, & a_n \\ f^{n+1}(u), & 0, & 1 \end{vmatrix} = 0 . \qquad (33)$$

This last determinant is, of course, nothing else than Lagrange's form

$$f^n x - f^n a_n = (x - a_n) f^{n+1}(u).$$

10. Regarding (23) as a series, and writing ρ for the ratio of convergency, we have

$$\rho = \frac{A_{r+1}}{A_r} = a_r - \frac{\frac{1}{2!} a_{r-1}^2 A_{r-1} - \frac{1}{3!} a_{r-2}^3 A_{r-2} + \ldots + (-1)^r \frac{a_0^{r+1} - x^{r+1}}{(r+1)!}}{a_{r-1} A_{r-1} - \frac{1}{2!} a_{r-2}^2 A_{r-2} + \ldots + (-1)^r \frac{a_0^r - x^r}{r!}},$$

$$= a_r - \tfrac{1}{2} a_{r-1} \frac{a_{r-1} A_{r-1} - \frac{1}{3} \frac{a_{r-2}^3}{a_{r-1}} A_{r-2} + \ldots + \frac{(-1)^r}{a_{r-1}} \frac{a_0^{r+1} - x^{r+1}}{(r+1)!}}{a_{r-1} A_{r-1} - \frac{1}{2!} a_{r-2}^2 A_{r-2} + \ldots + (-1)^r \frac{a_0^r - x^r}{r!}}. \qquad (34)$$

If now the quantities a_r and a_{r-1} are like signed, and numerically $a_r > a_{r-1}$ for all values of r, then the ratio in the second member is less than unity, when r is large enough. ρ is therefore less than a_r and greater than $a_r - \frac{1}{2} a_{r-1}$. In symbols

$$a_r > \rho > a_r - \tfrac{1}{2} a_{r-1}.$$

Hence, à priori,

$$a_r > \rho > \tfrac{1}{2} a_r.$$

It is therefore certain that the series will be absolutely convergent (the derivatives of fx being finite) if a_r be numerically less than unity. This does not assert that there are not other relations which may exist between the a's such that the formula may be written to infinity, but it does show that it is worthy of investigation for infinite series.

11. A_r is the value of y_r as obtained from the system of simultaneous equations

$$\begin{array}{l}
y = 1, \\
y + y_0 = 0, \\
xy + a_0 y_0 + y_1 = 0, \\
\frac{1}{2!} x^2 y + \frac{1}{2!} a_0^2 y_0 + a_1 y_1 + y_2 = 0, \\
\frac{1}{3!} x^3 y + \frac{1}{3!} a_0^3 y_0 + \frac{1}{2!} a_1^2 y_1 + a_2 y_2 + y_3 = 0, \\
\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}$$

That is

$$y_r = (-1)^{r+1} \begin{vmatrix} 1, & x\,, & \frac{1}{2!}x^2\,, & \dots \\ 1, & a_0, & \frac{1}{2!}a_0^2\,, & \dots \\ 0, & 1\,, & a_1\,, & \dots \\ 0, & 0\,, & 1\,, & \dots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} .$$

If n be infinite, this relation may be exhibited thus:

$$e^x - e^{a_0} + e^{a_1}A_1 - e^{a_2}A_2 + e^{a_3}A_3 - \dots = 0$$

$$\therefore \qquad e^{x-a_0} = 1 - e^{a_1-a_0}A_1 + e^{a_2-a_0}A_2 - \dots \tag{35}$$

We know that A_r is a rational integral function of x of the rth degree, which contains $x - a_0$ as a factor. But we have also learned that its first derivative contains the factor $x - a_1$, its pth derivative contains the factor $x - a_p$, and so on, until its $(r - 1)$th derivative contains $x - a_{r-1}$. If now these a's be all equal to a_0, then A_r must contain the factor $(x - a_0)^r$, and it cannot contain x in any other form. This being the case, it follows that the cofactor in A_r must be $(-1)^r 1/r!$, since

$$e^{x-a_0} = 1 - A_1 + A_2 - \dots$$

Therefore

$$A_r = (-1)^r \frac{(x - a_0)^r}{r!} ,$$

when all of the arbitraries are equal. This proof is not altogether satisfactory since it requires n to be infinite, before we know the series to be convergent.

Again, we know that

$$(-1)^r A_r = {}^{r}\mathrm{I}_{a_0}^{x} dx .$$

But if all of the a's are equal to a_0, then a_0 is an absolute constant, and no longer arbitrary. The operation I in this case degenerates into simple integration, and we have,* putting $x - a_0 = y$,

$$(-1)^r A_r = {}^{r}\mathrm{I}_{0}^{y} dy ,$$

$$= \frac{y^r}{r!} = \frac{(x - a_0)^r}{r!} . \tag{36}$$

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* In order to avoid dependence upon a symbolic operation which is not yet accepted, I shall give another proof of this result further on.

12. Examining the function A_r, we may regard it as a function of any of the quantities $a_0, \ldots, a_{r-1}$. It may in general be made to vanish by giving to any one of the arbitraries any one of the values which satisfy $A_r = 0$. a_{r-1} has one such value, a_{r-2} has two, a_{r-3} has three, ..., a_1 has $r - 1$ such values. We therefore have theoretically a choice of $\frac{1}{2}\, r(r - 1)$ values each of which when assigned to one of the arbitraries will cause A_r to disappear.

If all the a's down to a_r are equal to a_0, then $A_{r+1} = 0$ if

$$a_r = a_0 + \frac{x - a_0}{r + 1}.$$

For, by the above,

$$A_p = (-1)^p \frac{(x - a_0)^p}{p!}, \qquad (p \overset{=}{<} r),$$

and by (25)

$$A_{r+1} = a_r \frac{(x - a_0)^r}{r!} + \tfrac{1}{2!} a_0^2 \frac{(x - a_0)^{r-1}}{(r-1)!} + \tfrac{1}{3!} a_0^3 \frac{(x - a_0)^{r-2}}{(r-2)!} + \ldots - \frac{x^{r+1} - a_0^{r+1}}{(r+1)!}.$$

Putting $A_{r+1} = 0$, and noticing the identity

$$\frac{x^{r+1} - a_0^{r+1}}{(r+1)!} = \frac{(x - a_0)^{r+1}}{(r+1)!} + a_0 \frac{(x - a_0)^r}{r!} + \tfrac{1}{2!} a_0^2 \frac{(x - a_0)^{r-1}}{(r-1)!} + \ldots + \frac{a_0^{r-1}}{(r-1)!} \frac{(x - a_0)}{1!}.$$

we have

$$a_r \frac{(x - a_0)^r}{r!} = \frac{(x - a_0)^{r+1}}{(r + 1)!} + a_0 \frac{(x - a_0)^r}{r!}.$$

Whence

$$a_r = a_0 + \frac{x - a_0}{r + 1}. \tag{37}$$

Furthermore, consideration of (25) shows that this substitution completely eliminates a_r, and all terms involving a_r in the general formula disappear.

13. Heretofore we have considered the formula (22) as arranged according to the derivatives of the function. For some purposes an arrangement according to the powers of the variable is to be preferred.

Thus,

$$fx = C_0 - C_1 x + C_2 \frac{x^2}{2!} - \ldots + (-1)^n C_{n+1} \frac{x^{n+1}}{(n+1)!}, \tag{38}$$

wherein C_r is the determinant obtained by deleting the first row and $(r + 2)$th column in (22).

We have

$$C_0 = f^0 a_0 - \left| a_0 \right| f' a_1 + \begin{vmatrix} a_0, & \frac{1}{2!} a_0^2 \\ 1, & a_1 \end{vmatrix} f'' a_2 - \begin{vmatrix} a_0, & \frac{1}{2!} a_0^2, & \frac{1}{3!} a_0^3 \\ 1, & a_1, & \frac{1}{2!} a_1^2 \\ 0, & 1, & a_2 \end{vmatrix} f''' a_3 + \ldots$$

$$= f a_0 - \beta_1 f' a_1 + \beta_2 f'' a_2 - \ldots + (-1)^{n+1} \beta_{n+1} f^{n+1}(u).$$

These coefficients are connected by the recurrence formula

$$\beta_r = a_{r-1} \beta_{r-1} - \tfrac{1}{2!} a_{r-2}^2 \beta_{r-2} + \tfrac{1}{3!} a_{r-3}^3 \beta_{r-3} - \ldots + (-1)^{r+1} \frac{a_0^r}{r!} .$$

Again,

$$C_1 = -f' a_1 + \left| a_1 \right| f'' a_2 - \begin{vmatrix} a_1, & \frac{1}{2!} a_1^2 \\ 1, & a_2 \end{vmatrix} f''' a_3 + \begin{vmatrix} a_1, & \frac{1}{2!} a_1^2, & \frac{1}{3!} a_1^3 \\ 1, & a_2, & \frac{1}{2!} a_2^2 \\ 0, & 1, & a_3 \end{vmatrix} f^{iv} a_4 - \ldots$$

We may therefore write

$$C_1 x = - \mathbf{I}_0^x dx\, C_0 ;$$

and, generally,

$$(-1)^r C_r \frac{x^r}{r!} = {}^{r}\mathbf{I}_0^x dx\, C_0 .$$

We observe that,

$$C_n = (-1)^n f^n a_n + (-1)^{n+1} a_n f^{n+1}(u) ,$$

$$C_{n+1} = (-1)^{n+1} f^{n+1}(u) .$$

The series may then be written

$$fx = \sum_{i=0}^{i=n} {}^{i}\mathbf{I}_0^x dx\, C_0 + \frac{x^{n+1}}{(n+1)!} f^{n+1}(u) .$$

Where as before $\Sigma \mathrm{I}$ is the operator which turns C_0 into fx. If we examine the form of C_0 closely, we observe that it is neither more nor less than the value of fx determined from determinant (22), in which we put $x = 0$, or

$$C_0 = f(0) .$$

Therefore $\Sigma \mathrm{I}$ changes $f(0)$ into fx, and is again consistently the operator of enlargement. In the same way we are led through the operator to believe that

$$C_1 = f'(0) ;$$

and, on examination, we find this to be true.

The symbolic operator, therefore, leads us to the result

$$fx = \sum_{i=0}^{i=n} {}^{i}\!\int_0^x dx\,\boldsymbol{f}(0) + \frac{x^{n+1}}{(n+1)!} f^{n+1}(u)\,. \tag{39}$$

The form of this, which may be regarded as a generalized form of Maclaurin's theorem, may be written

$$fx = \boldsymbol{f}(0) + x\boldsymbol{f}'(0) + \frac{x^2}{2!}\boldsymbol{f}''(0) + \ldots + \frac{x^{n+1}}{(n+1)!} f^{n+1}(u)\,, \tag{40}$$

in which $\boldsymbol{f}^r(0)$ means that the determinant (22) is to be differentiated r times with respect to x, and then x put equal to zero, and from the result $\boldsymbol{f}^r(0)$ is to be obtained, which is written in bold-faced type in order to distinguish it from the ordinary meaning of $f^r(0)$. Or simply, $\boldsymbol{f}^r(0)$ is the rth derivative of the right hand member of (23), in which after differentiation x is put equal to zero. We see now that we might have written down Maclaurin's formula at once, and interpreted the coefficients from (23); but in doing so, we should have anticipated that result, and missed a beautiful verification of the preceding work. The result (40) enables us, then, to immediately arrange any series which has been obtained by (23) according to the powers of x.

14. Instead of writing out A_r according to its last column, as I preferred to do in (25), we may write it out according to x; thus,

$$A_r = M_0 - xM_1 + \frac{x^2}{2!}M_2 - \frac{x^3}{3!}M_3 + \ldots + (-1)^{r-1}\frac{x^{r-1}}{(r-1)!}a_{r-1} + (-1)^r\frac{x^r}{r!}\,,$$

$$= \sum_0^r (-1)^p M_p \frac{x^p}{p!}\,,$$

where

$$M_p = \begin{vmatrix} a_p\,, & \frac{1}{2!}a_p^2\,, & \ldots, & \dfrac{a_p^{r-p}}{(r-p)!} \\ 1\,, & a_{p+1}\,, & \ldots, & \dfrac{a_{p+1}^{r-p-1}}{(r-p-1)!} \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ 0\,, & 0\,, & \ldots,\; 1\,, & a_{r-1} \end{vmatrix}$$

$$= \sum_{m=1}^{m=r-p} (-1)^{m+1} M_{p+m}\frac{a_p^m}{m!}\,,$$

which may be expressed as the recurrence formula

$$M_p = a_{r-1}M_{p-1} - \frac{1}{2!}a_{r-2}^2 M_{p-2} + \ldots + (-1)^{r-p+1}\frac{a_p^{r-p}}{(r-p)!}$$

If all of the a's be equal to a, it is easy now to show, by induction, that

$$M_p = \frac{1}{(r-p)!} a^{r-p}.$$

Or otherwise, it may be easily shown* that M_p is the coefficient of x^{r-p} in the expansion of e^{ax}. Whence

$$A_r = \frac{a^r}{r!}\frac{x^0}{0!} - \frac{a^{r-1}}{(r-1)!}\frac{x^1}{1!} + \frac{a^{r-2}}{(r-2)!}\frac{x^2}{2!} - \dots + (-1)^{r-1}\frac{a^1}{1!}\frac{x^{r-1}}{(r-1)!}$$

$$+ (-1)^r \frac{a^0}{0!}\frac{x^r}{r!} = (-1)^r \frac{(x-a)^r}{r!},$$

which sustains the result reached in (36).

15. Let us stop here, for the present, the further analysis of the general formula, in order to notice a few of the special forms which flow from the general form through suppositions regarding the arbitrary constants.

1) As the simplest possible case, let $a_0 = a_1 = a_2 = \dots = a_n = 0$.

The formula then becomes, in virtue of (36),

$$fx = f(0) + xf'(0) + \dots + \frac{x^n}{n!} f^n(0) + \frac{x^{n+1}}{(n+1)!} f^{n+1}(u), \tag{41}$$

Maclaurin's formula, in which $(0 < u < x)$.

2) Let $a_0 = a_1 = \dots = a_n = a$.

We have again in virtue of (36),

$$fx = fa + (x-a)f'a + \frac{(x-a)^2}{2!} f''a + \dots$$

$$+ \frac{(x-a)^n}{n!} f^n a + \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}(u), \tag{42}$$

Taylor's formula, in which $(a < u < x)$.

3) Let $a_0 = 0$, and $a_1 = a_2 = \dots = a_n = x$.

Then in virtue of § 12, we have

$$fx = f(0) + xf'x + \dots + \frac{x^n}{n!} f^n x + \frac{x^{n+1}}{(n+1)!} f^{n+1}(u), \tag{43}$$

John Bernouilli's formula; $(0 < u < x)$.

4) Let $x - a = h$, $a = a_0$, and $a_1 = a_3 = a_5 = \dots = a + \frac{1}{2} h$.

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* See a note by Dr. Franklin in the American Journal of Math. Vol. VI, No. 3.

We know from (37) that the term in f'' will disappear; and it still further turns out that every alternate term after that vanishes, and we have, changing a to x,

$$f(x+h) = fx + hf'(x+\tfrac{1}{2}h) + \frac{1}{2^2}\frac{h^3}{3!}f'''(x+\tfrac{1}{2}h) + \dots$$

$$+ \frac{1}{2^{2n}}\frac{h^{2n+1}}{(2n+1)!}f^{2n+1}(x+\tfrac{1}{2}h) + \dots \tag{44}$$

This will be found useful, in the following form:

$$\frac{1}{h}\int_x^{x+h} fx\,dx = f(x+\tfrac{1}{2}h) + \tfrac{1}{3!}(\tfrac{1}{2}h)^2 f''(x+\tfrac{1}{2}h)$$

$$+ \tfrac{1}{5!}(\tfrac{1}{2}h)^4 f^{\text{iv}}(x+\tfrac{1}{2}h) + \dots \tag{45}$$

The series (44) was given by Dr McClintock in his Essay on the Calculus of Enlargement.*

In (44) put $x = 0$, then put $h = x$; whence

$$fx = f(0) + xf'(\tfrac{1}{2}x) + \frac{1}{2^2}\frac{x^3}{3!}f'''(\tfrac{1}{2}x) + \dots$$

$$+ \frac{1}{2^{2n}}\frac{h^{2n+1}}{(2n+1)!}f^{2n+1}(\tfrac{1}{2}x) + \dots, \tag{46}$$

a formula more convergent than Bernouilli's, and which we will find useful further on, in the form†

$$\frac{1}{x}\int_0^x fx\,dx = f(\tfrac{1}{2}x) + \frac{1}{3!}(\tfrac{1}{2}x)^2 f''(\tfrac{1}{2}x) + \dots$$

$$+ \frac{1}{(2n+1)!}(\tfrac{1}{2}x)^{2n} f^{2n}(\tfrac{1}{2}x) + \dots \tag{47}$$

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* See American Journal of Mathematics, Vol. II, No. 2, p. 122, where it is shown that this series may also be found by subtracting the development of $f(x+\frac{1}{2}h)$ by Taylor's formula, from that of $f(x-\frac{1}{2}h)$, and then writing $(x+\frac{1}{2}h)$ for x. It is also there shown how the series may be derived through the symbolic operation of the Calculus of Enlargement.

† This form was used by Fourier (Théorie de la Chaleur, p. 228. Paris, 1822) in his celebrated deduction of the expansion of an arbitrary function in terms of sines of the variable.

In (46) it may be shown that the general term is, if r be odd,

$$\frac{1}{2^{r-1}}\frac{x^r}{r!}\left[r+\frac{r(r-1)}{2\,!}+\frac{r(r-1)(r-2)}{3\,!}+\dots\right.$$
$$\left.+\frac{r(r-1)\dots(r-\frac{1}{2}(r+3))}{(\frac{1}{2}(r-1))\,!}-2^{r-1}\right]=\frac{1}{2^{r-1}}\frac{x^r}{r!};$$

and if r be even,

$$\frac{1}{2^{r-1}}\frac{x^r}{r!}\left\{\begin{array}{l} r+\dfrac{r(r-1)(r-2)}{3\,!}+\dfrac{r(r-1)(r-2)(r-3)(r-4)}{5\,!}+\dots \\ \quad+\dfrac{r\,!\,r\,!}{(\frac{1}{2}r)\,!\,(\frac{1}{2}r)\,!}+\dots+\dfrac{r(r-1)(r-2)}{3\,!}+r-2^{r-1}\end{array}\right\}=0.$$

5) Let $a_1=a+p^{-1}h$; then we have

$$f(x+h)=fx+hf'(x+p^{-1}h)+\tfrac{1}{2}p^{-1}(p-2)\,h^2f''(u)\,, \tag{48}$$

which has an interesting geometrical illustration.

If $a_0=0$, $a_1=p^{-1}x$, $a_2=q^{-1}x$; then

$$fx=f(0)+xf'(p^{-1}x)+\frac{p-2}{p}\frac{x^2}{2!}f''(q^{-1}x)$$
$$+\left[1-3\frac{p-2}{pq}-\frac{3}{p^2}\right]\frac{x^3}{3!}f'''(u)\,. \tag{49}$$

6) Let $x=y+nh$, $a_0=y$, $a_1=y+h$, $a_r=y+rh$; then we have the following formula:*

$$f(y+nh)=fy+nh\,f'(y+h)+n(n-2)\frac{h^2}{2!}f''(y+2h)+\dots$$
$$+n(n-r)^{r-1}\frac{h^r}{r\,!}f^{\,r}(x+rh)+\dots \tag{50}$$

And if $y=0$, and $nh=x$,

$$f(x)=f(0)+xf'\left[\frac{x}{n}\right]+\frac{n-2}{n}\frac{x^2}{2\,!}f''\left[\frac{2x}{n}\right]+\dots$$
$$+\left[\frac{n-r}{n}\right]^{r-1}\frac{x^r}{r!}f^r\left[\frac{r-1}{n}x\right]+\dots \tag{51}$$

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* This may be transformed into the series of Abel, which will be deduced further on. It is usually stated of Abel's series, that it is true for functions developable in powers of e^x. See Carr's Synopsis, p. 282. This point will be brought up again in Section III.

7) Let $x = y + nh$, $a_0 = y$, $a_1 = y - h$, $a_r = y - rh$; then

$$f(y + nh) = fy + nh\,f'(y - h) + n(n + 2)\frac{h}{2!}f''(y - 2h) + \dots$$
$$+ n(n + r)^{r-1}\frac{h^r}{r!}f^r(y - rh) + \dots \tag{52}$$

Compare the deduction of this, here given, with that given in Carr's Synopsis, p. 358, where it is obtained through the aid of symbolic operation; also, note there the misprint in the last term.

8) Let $x = y + h$, $a_0 = y$, $a_1 = y + h$, $a_r = y + rh$; then

$$f(x + h) = fy + hf'(y + h) - \tfrac{1}{2}h^2f''(y + 2h) + \tfrac{2}{3}h^3f'''(u), \tag{53}$$

and $y = 0$, $h = z$, gives

$$fz = f(0) + zf'(z) - \tfrac{1}{2}z^2f''(2z) + \tfrac{2}{3}z^3f'''(u). \tag{54}$$

These are particular cases of (50).

9) Let $x = y + h$, $a_0 = y$, $a_1 = y - h$, $a_r = y - rh$; then

$$f(x + h) = fy + hf'(y - h) + \tfrac{3}{2}h^2f''(y - 2h) + \tfrac{8}{3}h^3f'''(u), \tag{55}$$

and $y = 0$ gives

$$fh = f(0) + hf'(h) + \tfrac{3}{2}h^2f''(-2h) + \tfrac{8}{3}h^3f'''(u). \tag{56}$$

These interesting formulæ, (55) and (56), were given by Dr. McClintock in his Calculus of Enlargement, where they were doubtless given for the first time, deduced there by means of the symbolic operations, and possibly they have never been reached before in any other way. They are particular cases of (52), which may itself be got from (50) by changing the signs of n and h.

10) Other interesting forms may be got by combining the preceding formulæ. We notice the following:—

Add and subtract (50) and (52); thus,

$$f(y + nh) = fy + \tfrac{1}{2}nh\,[f'(y + h) + f'(y - h)] + \dots$$
$$+ \frac{n}{2}\frac{h^r}{r!}[(n - r)^{r-1}f^r(y + rh) + (n + r)^{r-1}f^r(y - rh)] + \dots, \tag{57}$$

and

$$-\int_{y-h}^{y+h} fx\,dx = \Sigma\frac{h^{r-1}}{r!}\left[(n-r)^{r-1}f^{r-1}(y+rh)-(n+r)^{r-1}f^{r-1}(y-rh)\right]. \quad (58)$$

In (44) change the sign of h, and add and subtract the result from (44). Whence,

$$y_m = m - \sum_1^{\infty}\frac{h^{2n-1}}{2^{2n-1}(2n-1)!}\left[f^{2n-1}(x+\tfrac{1}{2}h)-f^{2n-1}(x-\tfrac{1}{2}h)\right], \quad (59)$$

wherein y_m is the mid-ordinate of an arc $y = fx$ whose extreme ordinates are $2h$ apart, and m is the mid-ordinate of the chord of the arc.

Also

$$\int_{x-h}^{x+h} fx\,dx = \sum_0^{\infty}\frac{h^{2n+1}}{2^{2n}(2n+1)!}\left[f^{2n}(x+\tfrac{1}{2}h)+f^{2n}(x-\tfrac{1}{2}h)\right]. \quad (60)$$

Compare these last two equations with the theorems of Stirling and Boole.

11) For sake of future reference, we write down as an application of (46), in which we put $f = e^x$,

$$\frac{e^x-1}{xe^{\frac{1}{2}x}} = 1 + \frac{1}{3!}\left[\frac{x}{2}\right]^2 + \frac{1}{5!}\left[\frac{x}{2}\right]^4 + \ldots$$

12) In trying to get certain forms for a special purpose, I computed a number of partial series after the order of the Bernouilli formula, the first few terms of some of which I give here because their appearance is interesting. They all have reference to definite integration, and may be increased in number indefinitely. Possibly some very pretty series may be obtained in this way.

If, $a_0 = 0$, $a_1 = 0$, $a_2 = a_3 = \ldots = a_n = \frac{1}{3}x$, then

$$fx = f(0) + xf'(0) + \tfrac{1}{2!}x^2f''(\tfrac{1}{3}x) + \tfrac{1}{3}\cdot\tfrac{1}{4!}x^4f^{\text{iv}}(\tfrac{1}{3}x) + \tfrac{2}{27}\cdot\tfrac{1}{5!}x^5f^{\text{v}}(\tfrac{1}{3}x)$$
$$+ \tfrac{1}{9}\cdot\tfrac{1}{6!}x^6f^{\text{vi}}(u). \quad (62)$$

If, $a_0 = 0$, $a_1 = \frac{1}{2}x$, $a_2 = \frac{1}{3}x$, $a_3 = \frac{1}{4}x$, $a_4 = \frac{1}{5}x$, then

$$fx = f(0) + xf'(\tfrac{1}{2}x) + \tfrac{1}{4!}x^3f'''(\tfrac{1}{4}x) + \tfrac{1}{4}\cdot\tfrac{1}{4!}x^4f^{\text{iv}}(\tfrac{1}{5}x) + \tfrac{9}{32}\cdot\tfrac{1}{5!}x^5f^{\text{v}}(u). \quad (63)$$

If, $a_0 = 0$, $a_1 = \frac{1}{2}x$, $a_3 = \frac{1}{3}x$, $a_4 = \frac{1}{4}x$, then

$$fx = f(0) + xf'(\tfrac{1}{2}x) + \tfrac{1}{4!}x^3f'''(\tfrac{1}{3}x) + \tfrac{1}{3!}\tfrac{1}{4!}x^4f^{\text{iv}}(\tfrac{1}{4}x) + \tfrac{29}{144}\cdot\tfrac{1}{5!}x^5f^{\text{v}}(u). \quad (64)$$

If, $a_0 = 0$, $a_1 = 0$, $a_2 = \frac{2}{3}x$, $a_3 = \frac{1}{4}x$, then

$$fx = f(0) + xf'(0) + \tfrac{1}{2}x^2 f''(\tfrac{2}{3}x) + \tfrac{1}{3}x^3 f'''(\tfrac{1}{4}x) - \tfrac{1}{9}x^4 f^{\text{iv}}(u). \tag{65}$$

If, $a_0 = 0$, $a_1 = 0$, $a_2 = \frac{2}{3}x$, $a_3 = 5x/4\,!$, then

$$fx = f(0) + xf'(0) + \tfrac{1}{2}x^2 f''(\tfrac{2}{3}x) + \tfrac{1}{3}x^3 f'''(5x/4\,!) - \tfrac{3\,5\,4\,1}{4\,3\,2}\cdot\tfrac{1}{5!}x^5 f^{\text{v}}(u). \tag{66}$$

If, $a_0 = 0$, $a_1 = 0$, $a_2 = \frac{2}{3}x$, $a_3 = \frac{1}{3}x$, then

$$fx = f(0) + xf'(0) + \tfrac{1}{2}x^2 f''(\tfrac{2}{3}x) + \tfrac{1}{3}x^3 f'''(\tfrac{1}{3}x) - \tfrac{1\,4}{3}\cdot\tfrac{1}{4!}x^4 f^{\text{iv}}(u). \tag{67}$$

If, $a_0 = 0$, $a_1 = \frac{1}{2}x$, $a_2 = 2^{-2}x$, $a_3 = 2^{-3}x$, $a_4 = 2^{-4}x$, then

$$fx = f(0) + xf'(\tfrac{1}{2}x) + \tfrac{1}{3}\cdot\tfrac{1}{8}x^3 f'''(2^{-3}x) + \tfrac{1}{4}\cdot\tfrac{1}{16}x^4 f^{\text{iv}}(2^{-4}x) + \tfrac{1\,7}{3\,2}\cdot\tfrac{1}{5!}x^5 f^{\text{v}}(u). \tag{68}$$

13) Of particular interest are the logarithmic formulæ which flow from (22), and its special forms. The general formula comes directly from (23); thus,

$$\log x = \log a_0 - \frac{A_1}{a_1} - \frac{A_2}{a_2^2} - 2\,!\,\frac{A_3}{a_3^3} - \ldots - (n-1)\,!\,\frac{A_n}{a_n^n} - n\,!\,\frac{A_{n+1}}{u^{n+1}} \tag{69}$$

This is a formula which is exceedingly flexible. We may if desired put $a_0 = 1$ and thus eliminate $\log a_0$. The formula for $\log x$ contains in its first term one arbitrary, in its second two, and so on; so that in r terms we have at our disposal r arbitrary constants. We may thus derive a number of logarithmic formulæ by giving specific values to these arbitraries, knowing that we shall obtain convergent series under the conditions set forth in § 10, at least.

We notice only a few of the immediately apparent of these, rather more to verify the general formula than with any other object in view.

Of course, if $a_0 = a_1 = \ldots = a_n$,

$$\log(a_0^{-1}x) = a_0^{-1}(x - a_0) - \tfrac{1}{2}a_0^{-2}(x - a_0)^2 + \tfrac{1}{3}a_0^{-3}(x - a_0)^3 - \ldots, \tag{70}$$

in which $a_0 = 1$ gives

$$\log x = (x - 1) - \tfrac{1}{2}(x - 1)^2 + \tfrac{1}{3}(x - 1)^3 - \ldots, \tag{71}$$

and $x = 1$, $a_0 = y$ gives

$$\log y = (1 - y^{-1}) + \tfrac{1}{2}(1 - y^{-1})^2 + \ldots, \tag{72}$$

all well known forms.

$x = 1$ and $h = y - 1$ in (44) give at once the familiar

$$\tfrac{1}{2}\log y = h(h+2)^{-1} + \tfrac{1}{3}h^3(h+2)^{-3} + \tfrac{1}{5}h^5(h+2)^{-5} + \dots \qquad (73)$$

From (50), we get

$$\log(y+nh) = \log y + nh(y+h)^{-1} - \tfrac{1}{2}n(n-2)h^2(y+2h)^{-2} + \dots, \qquad (74)$$

or

$$\log(1+nhy^{-1}) = nh(y+h)^{-1} - \tfrac{1}{2}(1-2n^{-1})\,n^2h^2(y+2h)^{-2}$$
$$+ \tfrac{1}{3}(1-3n^{-1})^2\,n^3h^3(y+3h)^{-3} - \dots \qquad (75)$$

In this put $nhy^{-1} = z$,

$$\log(1+z) = z(1+n^{-1}z)^{-1} - \tfrac{1}{2}(1-2n^{-1})\,z^2(1+2n^{-1}z)^{-2}$$
$$+ \tfrac{1}{3}(1-3n^{-1})\,z^3(1+3n^{-1}z)^{-3} - \dots, \qquad (76)$$

or if $n^{-1} = a$,

$$\log(1+z) = z(1+az)^{-1} - 2^{-1}(1-2a)\,z^2(1+2az)^{-}$$
$$+ 3^{-2}(1-3a)^2\,z^3(1+3az)^{-3} - \dots \qquad (77)$$

$a = 0$ in this last gives Mercator's well known series

$$\log(1+z) = z - \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 - \dots$$

$n = 1$, in (76) gives

$$\log(1+z) = z(1+z)^{-1} - \tfrac{1}{2}z^2(1+2z)^{-2} + \tfrac{1}{3}2^2z^3(1+3z)^{-3}$$
$$- \tfrac{1}{4}3^3z^4(1+4z)^{-4} + \dots \qquad (78)$$

A curious formula is gotten by putting $n = z$ in (76), whence*

$$\log(1+z) = \frac{z}{2} - \frac{z(z-2)}{2\,.\,3^2} + \frac{z(z-3)^2}{3\,.\,4^3} - \frac{z(z-4)^3}{4\,.\,5^4} + \dots \qquad (79)$$

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* In connection with these logarithmic forms see Laurent, Traité d'Analyse, Tome III, p. 386; also Abel's works.

Writing of Abel's series, Laurent says "étudiée par M. Halphen, a été reconnue inexacte dans un grandes nombres de cas; en particulier, quand $\varphi(x) = \log x$, le second membre, loin de représentes $\log(a + x)$, représente une transcendante nouvelle. Quoi qu'il en soit, M. Bertrand démontre, dans son Traité de Calcul différentiel, que, si la fonction $\varphi(z)$ est dévelopable suivant les puissances de e^z, la formule aura lieu pour cette fonction."

16. Formula (22) may be exhibited in another form, which for some purposes is to be preferred to (22). Thus

$$\begin{vmatrix} fx, & 1, & (x-a_0), & \frac{(x-a_1)^2}{2!}, & \frac{(x-a_2)^3}{3!}, & \ldots, & \frac{(x-a_n)^{n+1}}{(n+1)!} \\ fx_0, & 1, & (x_0-a_0), & \frac{(x_0-a_1)^2}{2!}, & \frac{(x_0-a_2)^3}{3!}, & \ldots, & \frac{(x_0-a_n)^{n+1}}{(n+1)!} \\ f'x_1, & 0, & 1, & (x_1-a_1), & \frac{(x_1-a_2)^2}{2!}, & \ldots, & \frac{(x_1-a_n)^n}{n!} \\ f''x_2, & 0, & 0, & 1, & (x_2-a_2), & \ldots, & \frac{(x_2-a_n)^{n-1}}{(n-1)!} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f^nx_n, & 0, & 0, & 0, & 0, & \ldots, & 1, \ (x_n-a_n) \\ f^{n+1}(u), & 0, & 0, & 0, & 0, & \ldots, & 0, \ 1 \end{vmatrix} = 0. \qquad (80)$$

Let, $x_0 = a_0$, $x_1 = a_1$, $x_2 = a_2$, etc.; then

$$\begin{vmatrix} fx, & 1, & (x-a_0), & \frac{(x-a_1)^2}{2!}, & \frac{(x-a_2)^3}{3!}, & \ldots, & \frac{(x-a_n)^{n+1}}{(n+1)!} \\ fa_0, & 1, & 0, & \frac{(a_0-a_1)^2}{2!}, & \frac{(a_0-a_2)^3}{3!}, & \ldots, & \frac{(a_0-a_n)^{n+1}}{(n+1)!} \\ f'a_1, & 0, & 1, & 0, & \frac{(a_1-a_2)^2}{2!}, & \ldots, & \frac{(a_1-a_n)^n}{n!} \\ f''a_2, & 0, & 0, & 1, & 0, & \ldots, & \frac{(a_2-a_n)^{n-1}}{(n-1)!} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f^na_n, & 0, & 0, & 0, & 0, & \ldots, & 1, \ 0 \\ f^{n+1}(u), & 0, & 0, & 0, & 0, & \ldots, & 0, \ 1 \end{vmatrix} = 0. \qquad (81)$$

Expanding this with respect to the first column, we have

$$fx = fa_0 - A_1f'a_1 + A_2f'a_2 - \ldots + (-1)^n A_nf^na_n + \ldots, \qquad (82)$$

in which the *a*'s may be computed from the recurrence formula

$$A_r = \frac{(-1)^r}{r!}\left[(x-a_{r-1})^r - (a_0-a_{r-1})^r\right] + \frac{(-1)^r}{(r-1)!}(a_1-a_{r-1})^{r-1}A_1$$

$$+ \frac{(-1)^{r+1}}{(r-2)!}(a_2-a_{r-1})^{r-2}A_2 + \ldots + \tfrac{1}{3!}(a_{r-3}-a_{r-1})^3 A_{r-3}$$

$$- \tfrac{1}{2!}(a_{r-2}-a_{r-1})^2 A_{r-2}. \qquad (83)$$

In (82) and (83), make the substitutions

$$x - a_r = \sum_0^r b_r, \quad a_p - a_r = \sum_{p+1}^r b_r .$$

Then we get

$$fx = f(x - b_0) + C_1 f'(x - b_0 - b_1) + \ldots + C_n f^n(x - \sum_0^n b_r) + \ldots, \tag{84}$$

a formula published by Mr. Glashan as "An Extension of Taylor's Theorem," in the American Journal of Mathematics, Vol. I, No. 3.*

The coefficient C_r may be computed from (83) after making the substitutions, the first three are

$$1\,!\, C_1 = b_0 ,$$

$$2\,!\, C_2 = b_0^2 + 2b_0 b_1 ,$$

$$3\,!\, C_3 = b_0^3 + 3b_0^2 (b_1 + b_2) + 3b_0 (b_1^2 + 2b_1 b_2) .$$

Put $b_0 = a$, $b_1 = b_2 = \ldots = b_n = \beta$, and deduce as did Mr. Glashan, the series of Abel,

$$fx = f(x - a) + af'(x - a - \beta) + \frac{a(a + 2\beta)}{2\,!} f''[x - (a + 2\beta)] + \ldots . \tag{85}$$

The form in which Abel gives it, however, is†

$$f(x + a) = fx + af'(x + \beta) + \frac{a(a - 2\beta)}{2\,!} f''(x + 2\beta)$$

$$+ \frac{a(a - 3\beta)^2}{3\,!} f'''(x + 3\beta) + \ldots . \tag{86}$$

Compare this with series (50).

17. The celebrated Legendrian coefficient,‡

$$X_n = \frac{1}{2^n n\,!} \frac{d^n}{dx^n} (x^2 - 1)^n ,$$

is of the nth degree; it therefore belongs to the class of rational integral functions with which we are now dealing.

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[*] In a subsequent paper (Am. Jour. Math., Vol. IV, No. 3) Mr. Glashan remarks that this series is due to Professor Cayley, he having given a proof of it in *Solutions of the Cambridge Senate-House Problems*, 1848–51, pp. 94–96.

[†] See Oeuvres Complètes de Niels Abel, Tome II, p. 73, where Abel deduces this expansion by means of generating functions.

[‡] Called also Laplace's coefficient of a single variable, also Kugelfunctionen, fonctions sphériques, and spherical harmonics. See Todhunter's Treatise on Laplace's, Lamé's, and Bessel's Functions.

Let $X^r_{n,a}$ be the rth derivative of the nth Legendrian, in which after differentiation we replace x by the arbitrary constant a. Then, we have

$$\begin{vmatrix} fx & , 1, & X_1 & , X_2 & , \dots, & X_{n+1} \\ fy & , 1, & X_{1,y}, & X_{2,y}, & \dots, & X_{n+1,y} \\ f'a & , 0, & X'_{1,a}, & X'_{2,a}, & \dots, & X'_{n+1,a} \\ f''b & , 0, & 0 & , X''_{2,b}, & \dots, & X''_{n+1,b} \\ f'''c & , 0, & 0 & , 0 & , \dots, & X'''_{n+1,c} \\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdot \\ f^nz & , 0, & 0 & , 0 & , \dots, & X^n_{n+1,z} \\ f^{n+1}(u), & 0, & 0 & , 0 & , \dots, & 1 \end{vmatrix} = 0. \qquad (87)$$

We notice particularly the case of the general formula, wherein

$$y = a = b = c = \dots = z = 0.$$

Use, after McClintock, the symbols $x_b^{)n}$ and $x_b^{n(}$ to denote the upper and lower factorial products respectively

$$x(x+b)(x+2b)\dots(x+\overline{n-1}.b),$$

$$x(x-b)(x-2b)\dots(x-\overline{n-1}.b).$$

Make the above substitution in (87), and consider the nth Legendrian. Differentiate it n times and put $x = 0$ in the results, beginning with the function itself. These are, in order, the constituents of the nth column. After factoring out the diagonal constituent, the nth column, from the diagonal up, will be found to be

$$1, 0, -\frac{1}{2_2^{)1}(2n-1)_2^{1(}}, 0, \frac{1}{2_2^{)2}(2n-1)_2^{2(}}, 0, -\frac{1}{2_2^{)3}(2n-1)_2^{3(}}, \dots, \frac{2^{n-1}(n-1)!}{(2n-1)!}X_n.$$

If n be even, the next to the last term is

$$\frac{(-1)^{\frac{n}{2}}}{2_2^{)\frac{n}{2}}(2n-1)_2^{\frac{n}{2}(}}.$$

If n be odd, the next to the last term is zero, and the next to the last but one is

$$\frac{(-1)^{\frac{n-1}{2}}}{2_2^{)\frac{n-1}{2}}(2n-1)_2^{\frac{n-1}{2}(}}.$$

Therefore, we have

$$\begin{vmatrix} fx, & 1, & X_1, & \frac{X_2}{3}, & \frac{1}{3\,.\,5}X_3, & \frac{1}{3\,.\,5\,.\,7}X_4, & \dots, & \frac{2^n n!\,X_{n+1}}{(2n+1)!} \\ f0, & 1, & 0, & \frac{-1}{2\,.\,3}, & 0, & \frac{1}{2\,.\,4\,.\,5\,.\,7}, & \dots, & \dots \\ f'0, & 0, & 1, & 0, & \frac{-1}{2\,.\,5}, & 0, & \dots, & \dots \\ f''0, & 0, & 0, & 1, & 0, & \frac{-1}{2\,.\,7}, & \dots, & \dots \\ f'''0, & 0, & 0, & 0, & 1, & 0, & \dots, & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f^n0, & 0, & 0, & 0, & 0, & 0, & \dots, & 0 \\ f^{n+1}(u), & 0, & 0, & 0, & 0, & 0, & \dots, & 1 \end{vmatrix} = 0. \qquad (88)$$

Write out (88) with respect to the first column, getting

$$fx = f0 - C_1 f'0 + C_2 f''0 - \dots + (-1)^n C_n f^n 0 + \dots + R, \qquad (89)$$

in which we have for the coefficient C_r

$$C_r = \begin{vmatrix} 1, & X_1, & \frac{X_2}{3}, & \frac{1}{3\,.\,5}X_3, & \dots, & \frac{2^{r-1}(r-1)!}{(2r-1)!}X_r \\ 1, & 0, & \frac{-1}{2\,.\,3}, & 0, & \dots, & \dots \\ 0, & 1, & 0, & \frac{-1}{2\,.\,5}, & \dots, & \dots \\ 0, & 0, & 1, & 0, & \dots, & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & 0, & 0, & \dots, 1, & 0 \end{vmatrix}, \qquad (90)$$

in which the last column is to be filled in, as the general column is formed above, according as r is even or odd. The above value of C_r writes out in the recurrence formula,

$$C_r = \frac{C_{r-2}}{2_2^{)1}(2r-1)_2^{1(}} - \dots + \frac{(-1)^{m+1}C_{r-2m}}{2_2^{)m}(2r-1)_2^{m(}} + \dots$$

$$+ (-1)^r \frac{2^{r-1}(r-1)!\,X_r}{(2r-1)!}. \qquad (91)$$

It is easy to see that if r is odd, C_r contains no even Legendrians; and if even, no odd ones.

We have

$$C_0 = 1\,,$$

$$C_1 = -x\,,$$

$$C_2 = \frac{1}{3}X_2 + \frac{1}{2\,.\,3} = \frac{1}{2!}x^2\,,$$

$$C_3 = -\frac{1}{3\,.\,5}X_3 + \frac{1}{2\,.\,3}C_1 = -\frac{1}{3!}x^3\,,$$

$$C_4 = \frac{1}{3\,.\,5\,.\,7}X_4 + \frac{1}{2\,.\,7}C_2 - \frac{1}{2\,.\,4\,.\,5\,.\,7} = \frac{1}{4!}x^4\,,$$

$$.\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\;;$$

and we may show by induction that $C_r = (-1)^r x^r/r!$; so that (89) brings us back again to Maclaurin's form. Equating the determinant (90) to $(-1)^r x^r/r!$, we have an expression for any integral power of x in terms of Legendrians, which when written out gives

$$(-1)^r x^r/r! = a_r X_r + a_{r-2}X_{r-2} + a_{r-4}X_{r-4} + \ldots\,,$$

where the a's are the minors of the Legendrians in the first row. Since we know (see Todhunter's Functions, p. 15) that

$$(-1)^m \frac{2^{m-1}}{m(2m-1)!}\,a_m = \frac{2m+1}{2}\int_{-1}^{+1} X_m\, x^r dx\,,$$

the above gives the value of this integral expressed as a determinant.

While the preceding result is interesting, it is not what we wish, as our object is to obtain the expansion of an arbitrary function in terms of Legendre's coefficients. To effect this we proceed to change the determinant (88) into a shape which we shall frequently employ hereafter. In the first place for the sake of brevity in printing, let us temporarily put

$$r_p = \frac{1}{2_2^{)p} r_2^{p(}}\,.$$

Treat the determinant (88) according to the method of (13), and consider Fx. Run the first column and row to the middle, the odd columns to the right, the even ones to the left, the odd rows to the bottom, and the even rows to the top, thus obtaining for Fx the determinant

$$\left.\begin{array}{cccccccccccc}
1, & -5_1, & +9_2 & ,\ldots, & \pm(2n-3)_{n-1} & ,f'0 & ,0, & 0 & , & 0 & ,\ldots, & 0 \\
0, & 1, & -9_1 & ,\ldots, & \pm(2n-3)_{n-2} & ,f'''0 & ,0, & 0 & , & 0 & ,\ldots, & 0 \\
0, & 0, & 1 & ,\ldots, & \pm(2n-3)_{n-3} & ,f^{\mathrm{v}}0 & ,0, & 0 & , & 0 & ,\ldots, & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0, & 0, & 0 & ,\ldots, & 1 & ,f^{2n-1}0, & 0, & 0 & , & 0 & ,\ldots, & 0 \\
X_1, & \frac{1}{3.5}X_3, & \frac{1}{3.5.7.9}X_5 & ,\ldots, & \frac{2^{2n-3}(2n-3)!}{(4n-3)!}X_{2n-1} & ,fx & ,\frac{1}{3}X_2, & \frac{1}{3.5.7}X_4 & , & \frac{1}{3.5.7.9.11}X_6 & ,\ldots, & \frac{2^{2n}(2n)!}{(4n-1)!}X_{2n} \\
0, & 0, & 0 & ,\ldots, & 0 & ,f''0 & ,1, & -7_1 & , & 11_2 & ,\ldots, & \pm(2n-1)_{n-1} \\
0, & 0, & 0 & ,\ldots, & 0 & ,f^{\mathrm{iv}}0 & ,0, & 1 & , & -11_1 & ,\ldots, & \pm(2n-1)_{n-2} \\
0, & 0, & 0 & ,\ldots, & 0 & ,f^{\mathrm{vi}}0 & ,0, & 0 & , & 1 & ,\ldots, & \pm(2n-1)_{n-3} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0, & 0, & 0 & ,\ldots, & 0 & ,f^{2n}0 & ,0, & 0 & , & 0 & ,\ldots & 1
\end{array}\right|$$

By aid of this we write

$$fx = A_0 - A_1X_1 + \frac{1}{3}A_2X_2 - \frac{1}{3.5}A_3X_3 + \ldots$$

$$+ (-1)^n\frac{2^{n-1}(n-1)!}{(2n-1)!}A_nX_n + R. \tag{93}$$

In which we have for the coefficients A_r,

$$(-1)^rA_r = \left|\begin{array}{ccccc}
-(2r+3)_1, & +(2r+5)_2, & -(2r+7)_3, & \ldots, & f^r0 \\
1 & ,-(2r+5)_1, & +(2r+7)_2, & \ldots, & f^{r+2}0 \\
0 & ,\quad 1 & ,-(2r+7)_1, & \ldots, & f^{r+4}0 \\
0 & ,\quad 0 & ,\quad 1 & ,\ldots, & f^{r+6}0 \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
0 & ,\quad 0 & ,\quad 0 & ,\ldots, & 1, f^n0
\end{array}\right|. \tag{94}$$

To determine A_0, which is the Fy of (13), we merely put $x=0$ in (93); whence

$$A_0 = f0 - \frac{1}{2.3}A_2 - \frac{1}{2_2^{(2)}7_2^{(2)}}A_4 - \frac{1}{2_2^{(3)}11_2^{(3)}}A_6 - \ldots - \frac{1}{2_2^{(n)}(2n-1)_2^{(n)}}A_{2n} - \ldots,$$

in which the A's are the same as given by (94).

We notice, in connection with the first expansion, that the minor of $f^{2n}0$ in the determinant (92) is the equivalent of $x^{2n}/(2n)!$, and the minor of $f^{2n-1}0$ is the value of $x^{2n-1}/(2n-1)!$ (See Todhunter's Functions).

18. Consider the Hypergeometric series

$$1-\frac{\alpha_1\beta_1}{\gamma_1}\frac{x}{1!}+\frac{\alpha_1^{1(}\beta_1^{)1}}{\gamma_1^{)1}}\frac{x^2}{2!}-\ldots+(-1)^{m+1}\frac{\alpha_1^{m(}\beta_1^{)m}}{\gamma_1^{)m}}\frac{x^{m+1}}{(m+1)!}+\ldots,$$

which is always convergent for $x<1$.

For brevity put $\rho^{)m}=\beta_1^{)m}/\gamma_1^{)m}$. Give α the values 1, 2, 3, etc., in succession, thus forming the functions,

$$H_1=1-\rho x\,,$$

$$H_2=1-2\rho x+2!\,\rho^{)1}\frac{x^2}{2!}\,,$$

$$H_3=1-3\rho x+3^{1(}\rho^{)1}\frac{x^2}{2!}-3!\,\rho^{)2}\frac{x^3}{3!}\,,$$

. . . .

$$H_r=1-r\rho x+r^{1(}\rho^{)1}\frac{x^2}{2!}-\ldots+(-1)^{m+1}r^{m(}\rho^{)m}\frac{x^{m+1}}{(m+1)!}+\ldots+r!\,\rho^{)r-1}\frac{x^r}{r!}\,.$$

Substitute the functions $H_r/r!$ in the composite, and put all of the arbitraries equal to zero, and remove the common factors from the rows; whence

$$\begin{vmatrix}
fx\,, & 1\,, & H_1, & \frac{H_2}{2!}\,, & \frac{H_3}{3!}\,, & \frac{H_4}{4!}\,, & \ldots, & \frac{H_{n+1}}{(n+1)!} \\
f0\,, & 1\,, & 1\,, & \frac{1}{2!}\,, & \frac{1}{3!}\,, & \frac{1}{4!}\,, & \ldots, & \frac{1}{(n+1)!} \\
-\frac{f'0}{\rho}\,, & 0\,, & 1\,, & 1\,, & \frac{1}{2!}\,, & \frac{1}{3!}\,, & \ldots, & \frac{1}{n!} \\
+\frac{f''0}{\rho^{)1}}\,, & 0\,, & 0\,, & 1\,, & 1\,, & \frac{1}{2!}\,, & \ldots, & \frac{1}{(n-1)!} \\
-\frac{f'''0}{\rho^{)2}}\,, & 0\,, & 0\,, & 0\,, & 1\,, & 1\,, & \ldots, & \frac{1}{(n-2)!} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\frac{(-1)^n}{\rho^{)n-1}}f^n0, & 0\,, & 0\,, & 0\,, & 0\,, & 0\,, & \ldots, 1, & 1 \\
f^{n+1}(u)\,, & 0\,, & 0\,, & 0\,, & 0\,, & 0\,, & \ldots, 0, & 1
\end{vmatrix}=0\,. \qquad (95)$$

Therefore,

$$fx=A_0-A_1H_1+\frac{1}{2!}A_2H_2-\frac{1}{3!}A_3H_3+\ldots+(-1)^n\frac{1}{n!}A_nH_n+R, \quad (96)$$

wherein

$$(-1)^r A_r = \begin{vmatrix} +\dfrac{f^r0}{\rho^{)r-1}} & ,1, & \dfrac{1}{2!}, & \dfrac{1}{3!}, & \dfrac{1}{4!}, & \ldots, & \dfrac{1}{n!} \\ -\dfrac{f^{r+1}0}{\rho^{)r}} & ,1, & 1\ , & \dfrac{1}{2!}, & \dfrac{1}{3!}, & \ldots, & \dfrac{1}{(n-1)!} \\ +\dfrac{f^{r+2}0}{\rho^{)r+1}} & ,0, & 1\ , & 1\ , & \dfrac{1}{2!}, & \ldots, & \dfrac{1}{(n-2)!} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \dfrac{(-1)^n}{\rho^{)r+n-1}} f^{r+n}0, & 0\ , & 0\ , & 0\ , & 0\ , & \ldots, & 1 \end{vmatrix}$$

$$= \frac{1}{\rho^{)r-1}} f^r0 + \frac{1}{\rho^{)r}} f^{r+1}0 + \frac{1}{2!\,\rho^{)r+1}} f^{r+2}0 + \ldots + \frac{1}{n!\,\rho^{)r+n-1}} f^{r+n}0,$$

in virtue of (36). To get A_0, we make $x = 0$ in (96), noticing that all of the H's then become unity; whence

$$A_0 = f0 + A_1 - \frac{1}{2!} A_2 + \ldots + (-1)^{n+1} \frac{1}{n!} A_n + \ldots$$

18. Consider Bessel's Function

$$J_n(x) = \frac{x^n}{2^n n!} \left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2\,.\,4(2n+2)(2n+4)} - \ldots \right],$$

$$= \sum_{r=0}^{r=\infty} \frac{(-1)^r}{r!\,(n+r)!} \left[\frac{x}{2} \right]^{n+2r},$$

a series always convergent.

We have

$$\begin{vmatrix} fx, & J_0(x), & J_1(x), & J_2(x), & J_3(x), & J_4(x), & J_5(x), & \ldots, & J_{n+1}(x) \\ f0, & 1, & 0, & 0, & 0, & 0, & 0, & \ldots, & 0 \\ 2^1 f'0, & 0, & 1, & 0, & 0, & 0, & 0, & \ldots, & 0 \\ 2^2 f''0, & \dfrac{-2!}{1!\,1!}, & 0, & 1, & 0, & 0, & 0, & \ldots, & 0 \\ 2^3 f'''0, & 0, & \dfrac{-3!}{1!\,2!}, & 0, & 1, & 0, & 0, & \ldots, & 0 \\ 2^4 f^{\text{iv}}0, & \dfrac{+4!}{2!\,2!}, & 0, & \dfrac{-4!}{1!\,3!}, & 0, & 1, & 0, & \ldots, & 0 \\ 2^5 f^{\text{v}}0, & 0, & \dfrac{+5!}{2!\,3!}, & 0, & \dfrac{-5!}{1!\,4!}, & 0, & 1, & \ldots, & 0 \\ 2^6 f^{\text{vi}}0, & \dfrac{-6!}{3!\,3!}, & 0, & \dfrac{+6!}{2!\,4!}, & 0, & \dfrac{-6!}{1!\,5!}, & 0, & \ldots, & 0 \\ 2^7 f^{\text{vii}}0, & 0, & \dfrac{-7!}{3!\,4!}, & 0, & \dfrac{+7!}{2!\,5!}, & 0, & \dfrac{-7!}{1!\,6!}, & \ldots, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f^{n+1}(u), & 0, & 0, & 0, & 0, & 0, & 0, & \ldots, & 1 \end{vmatrix} = 0. \qquad (97)$$

The $(n + 2)$th row being

$$2^n f^n 0, 0, \frac{\pm n!}{(\frac{1}{2}n - \frac{1}{2})!\,(\frac{1}{2}n + \frac{1}{2})!}, 0, \ldots, 0, -C_{n,3}, 0, +C_{n,2}, 0, -C_{n,1}, 0, 1, 0, 0 \ldots,$$

or

$$2^n f^n 0, 0, \frac{\pm n!}{(\frac{1}{2}n)!\,(\frac{1}{2}n)!}, 0, \ldots, 0, -C_{n,3}, 0, +C_{n,2}, 0, -C_{n,1}, 0, 1, 0, 0, \ldots,$$

according as n is odd or even respectively. $C_{n,r}$ meaning the number of combinations of n things taken r at a time, or simply the binomial coefficients.

Expand (97) according to form (13), and arrange Fx thus

$$\left|\begin{array}{ccccccccccccc}
1 & 0 & 0 & 0 & 0 & \ldots & 2^1 f'0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
-C_{3,1} & 1 & 0 & 0 & 0 & \ldots & 2^3 f'''0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
+C_{5,2} & -C_{5,1} & 1 & 0 & 0 & \ldots & 2^5 f^{v}0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
-C_{7,3} & +C_{7,2} & -C_{7,1} & 1 & 0 & \ldots & 2^7 f^{vii}0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
+C_{9,4} & -C_{9,3} & +C_{9,2} & -C_{9,1} & 1 & \ldots & 2^9 f^{ix}0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
J_1(x) & J_3(x) & J_5(x) & J_7(x) & J_9(x) & \ldots & fx & J_0(x) & J_2(x) & J_4(x) & J_6(x) & J_8(x) & \ldots \\
0 & 0 & 0 & 0 & 0 & \ldots & f0 & 1 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & \ldots & 2^2 f''0 & -C_{2,1} & 1 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & \ldots & 2^4 f^{iv}0 & +C_{4,2} & -C_{4,1} & 1 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & \ldots & 2^6 f^{vi}0 & -C_{6,3} & +C_{6,2} & -C_{6,1} & 1 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & \ldots & 2^8 f^{viii}0 & +C_{8,4} & -C_{8,3} & +C_{8\ 2} & -C_{8,1} & 1 & \ldots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right| . \tag{98}$$

Whence we obtain, expanding with respect to the middle row,

$$fx = A_0 J_0(x) - A_1 J_1(x) + A_2 J_2(x) - \ldots + (-1)^n A_n J_n(x) + R, \tag{99}$$

in which we have

$$A_r = \left|\begin{array}{cccccc}
f0, & 1, & 0, & \ldots, & & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
2^r f^r 0, & \ldots & \ldots & \ldots & \ldots, & \dfrac{-r!}{1!\,(r-1)!}
\end{array}\right|$$

$$= -C_{r,1} A_{r-2} - C_{r,2} A_{r-4} - \ldots - C_{r,m} A_{r-2m} - \ldots + (-1)^r 2^r f^r 0.$$

The series may be verified to any extent by arranging according to powers of x, and thus identifying by means of Maclaurin's form. We know (see Todhunter's Functions) that any integral power of x may be expressed in terms of Besselian functions; we may therefore expect to find $x^r/r!$ equivalent to the coefficient of f^r0, in the expansion of (98) with respect to its middle column, which is expressed in terms of an infinite series of Besselian functions written in the form of a determinant. Expanding (98) with respect to the middle column, we get

$$fx = A_0 f0 - 2'A_1 f'0 + 2^2 A_2 f''0 + \dots + (-1)^r 2^r A_r f^r0 + R\,, \quad (100)$$

wherein

$$(-1)^r A_r = \begin{vmatrix} J_r(x)\ , & -C_{r+2,1}, & +C_{r+4,2}, & -C_{r+6,3}, & \dots \\ J_{r+2}(x), & 1\ , & -C_{r+4,1}, & +C_{r+6,2}, & \dots \\ J_{r+4}(x), & 0\ , & 1\ , & -C_{r+6,1}, & \dots \\ J_{r+6}(x), & 0\ , & 0\ , & 1\ , & \dots \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \end{vmatrix}.$$

Further discussion of the expansion of fx in terms of Bessel's functions is reserved for later consideration, when a different method of treatment will be applied.

19. This section, while treating rational integral functions as the type of those functions which yield a body-determinant unity, includes also certain transcendental functions whose coefficients may be put in recurrence formulæ. Examples of these will be given below.

20. To expand fx in terms of the central-factorial functions

$$\begin{aligned} x^{)2r+1(} &\equiv (x - ar)\dots(x - a)\,x\,(x + a)\dots(x + ra)\,, \\ &\equiv x\,(x^2 - a^2)\,(x^2 - 2^2a^2)\dots(x^2 - r^2a^2)\,, \\ &\equiv x^{2r+1} - a^2P_{r,1}\,x^{2r-1} + a^4P_{r,2}\,x^{2r-2} - \dots + (-1)^{r+1}\,a^{2r-2}P_{r,r-1}x^3 \\ &\quad + (-1)^r\,a^{2r}(r\,!)^2x\,, \end{aligned}$$

wherein $P_{r,p}$ is used to represent the sum of the products, without repetition, of the quantities $1, 2^2, \dots, r^2$ taken p at a time, and a is an arbitrary constant.

Substitute in the composite the functions

$$\frac{1}{(2r+1)\,!}\,x^{)2r+1(},$$

and put $x = 0$ after differentiation. Remembering in virtue of theorem § 4 that only the odd derivative rows will appear, we have

$$\left|\begin{array}{lllllllll} fx & ,\ 1, & x, & \frac{1}{3!}x^{)3(}, & \frac{1}{5!}x^{)5(}, & \frac{1}{7!}x^{)7(} & ,\ \ldots, & \ldots, & \ldots \\ f0 & ,\ 1, & 0, & 0\ , & 0\ , & 0 & ,\ \ldots, & \ldots, & \ldots \\ f'0 & ,\ 0, & 1, & -\frac{1}{6}a^2, & +\frac{1}{30}a^4, & -\frac{1}{140}a^6, & \ldots, & \ldots, & \ldots \\ f'''0 & ,\ 0, & 0, & 1\ , & -\frac{1}{4}a^2\ , & +\frac{7}{120}a^4, & \ldots, & \ldots, & \ldots \\ f^{\mathrm{v}}0 & ,\ 0, & 0, & 0\ , & 1\ , & -\frac{1}{8}a^2 & ,\ \ldots, & \ldots, & \ldots \\ f^{\mathrm{vii}}0 & ,\ 0, & 0, & 0\ , & 0\ , & 1 & ,\ \ldots, & \ldots, & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f^{2n-1}0, & 0, & 0, & 0\ , & 0\ , & 0 & ,\ \ldots, & 1\ , & \ldots \\ \Phi(u) & ,\ 0, & 0, & 0\ , & 0\ , & 0 & ,\ \ldots, & 0\ , & 1 \end{array}\right| = 0, \quad (101)$$

wherein the column headed with $x^{)2r+1(}$ is, from the diagonal up,

$$1, -\frac{(2r-1)!}{(2r+1)!}a^2P_{r,1}, +\frac{(2r-3)!}{(2r+1)!}a^4P_{r,2}, \ldots, \frac{(-1)^{r+1}3!}{(2r+1)!}a^{2r-2}P_{r,r-1},$$
$$\frac{(-1)^r a^{2r}(r!)^2}{(2r+1)!}, 0, \frac{x^{)2r+1(}}{(2r+1)!}.$$

If we expand with respect to the first column, we express the coefficient of $f^{(2p-1)}0$ in finite terms of the factorial functions. If we expand with respect to the factorial functions, we derive the formula

$$fx = f0 + A_1 x + A_3\frac{x^{)3(}}{3!} + \ldots + A_{2n+1}\frac{x^{)2n+1(}}{(2n+1)!} + \ldots, \qquad (102)$$

wherein the coefficient A_{2r+1} is an infinite series of the form

$$A_{2r+1} = C_1 f^{2r+1}(0) + C_2 f^{2r+3}(0) + \ldots.$$

The coefficients C being numbers which may be computed directly from the determinant, and may be connected by a recurrence formula.

The function fx should, in general, be an odd function.

21. In like manner we may consider the expansion of fx in terms of the upper factorial functions

$$x^{)r} = x(x+a)(x+2a)\ldots(x+\overline{r-1}a),$$
$$= x^r + P_{r,1}ax^{r-1} + P_{r,2}a^2x^{r-2} + \ldots + P_{r,r-1}a^{r-1}x^2 + a^r(r-1)!\,x,$$

(which becomes the lower factorial $x^{r(}$ if a is negative) wherein now $P_{r,p}$ is the sum of the products, without repetition, of the quantities $1, 2, \ldots, r-1$ taken p at a time.

Substituting in the composite the functions

$$\frac{1}{r!}x^{)},$$

since

$$\left[\frac{d^p x^{)r}}{dx^p}\right]_{x=0} = a^{r-p}p!\,P_{r,r-p},$$

we have

$$\begin{vmatrix} fx\ , & 1\ , & x\ , & \frac{1}{2!}x^{)2}, & \frac{1}{3!}x^{)3}, & \frac{1}{4!}x^{)4}\ , & \ldots\ , & \ldots \\ f0\ , & 1\ , & 0\ , & 0\ , & 0\ , & 0\ , & \ldots\ , & \ldots \\ f'0\ , & 0\ , & 1\ , & \frac{1}{2}a\ , & \frac{1}{3}a^2, & \frac{1}{4}a^3\ , & \ldots\ , & \ldots \\ f''0\ , & 0\ , & 0\ , & 1\ , & a\ , & \frac{11}{12}a^2, & \ldots\ , & \ldots \\ f'''0\ , & 0\ , & 0\ , & 0\ , & 1\ , & \frac{3}{2}a\ , & \ldots\ , & \ldots \\ f^{\text{iv}}0\ , & 0\ , & 0\ , & 0\ , & 0\ , & 1\ , & \ldots\ , & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \Phi(u)\ , & 0\ , & 0\ , & 0\ , & 0\ , & 0\ , & \ldots\ , & 1 \end{vmatrix} = 0. \tag{103}$$

The column headed with $x^{)r}$ being

$$1\,,\ \frac{r-1}{2}a\,,\ \frac{(r-2)!}{r!}a^2P_{r.2}\,,\ \ldots,\ \frac{2!}{r!}a^{r-1}P_{r,r-1}\,,\ \frac{a^r}{r!}\,.$$

We have from (103)

$$fx = f0 + \sum_1^n A_r\frac{x^{)r}}{r!}, \tag{104}$$

the same remarks applying to the coefficient A_r as in the preceding article.

If we put $a = 0$, we have, of course, Maclaurin's formula. These interesting factorial series will be taken up again later on.

22. We close the section with a few examples of transcendental function which may be given as illustrations of the application of the method of section II to this class of functions.

We know that

$$(\sin^{-1}ax)^n = (ax)^n + A_2\,(ax)^{n+2} + A_4\,(ax)^{n+4} + \ldots,$$

where the coefficients A are numbers to be obtained by equating the coefficients of c in the two series

$$e^{c\sin^{-1}ax} = 1 + cax + \frac{(cax)^2}{2!} + \frac{c(c^2+1)}{3!}(ax)^3 + \frac{c^2(c^2+2^2)}{4!}(ax)^4 + \ldots,$$

$$e^{c\sin^{-1}ax} = ax + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \ldots.$$

Making the substitutions, and putting $x = 0$ after differentiation, we have, letting $\theta_x^n = (\sin^{-1}ax)^n$,

$$\begin{vmatrix}
fx, & 1, & \theta_x, & \theta_x, & \theta_x^3, & \theta_x^4, & \ldots, & \theta_x^{n+1} \\
fy, & 1, & \theta_y, & \theta_y^2, & \theta_y, & \theta_y^4, & \ldots, & \theta_y^{n+1} \\
f'0, & 0, & 1, & 0, & 0, & 0, & \ldots, & 0 \\
\frac{f''0}{2!}, & 0, & 0, & 1, & 0, & 0, & \ldots, & 0 \\
\frac{f'''0}{3!}, & 0, & \frac{1}{2.3}, & 0, & 1, & 0, & \ldots, & 0 \\
\frac{f^{\mathrm{iv}}0}{4!}, & 0, & 0 & \frac{2^2}{3.4}, & 0, & 1, & \ldots, & 0 \\
\frac{f^{\mathrm{v}}0}{5!}, & 0, & \frac{1.3}{2.4.5}, & 0, & \frac{3^2+1}{4.5}, & 0, & \ldots, & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\Phi(u), & 0, & 0, & 0, & 0, & 0, & \ldots, & 1
\end{vmatrix} = 0. \qquad (105)$$

Therefore

$$fx - fy = \sum_{r=1}^{r=n} A_r[(\sin^{-1}ax)^r - (\sin^{-1}ay)^r]$$

wherein we have for the first five values of A,

$$A_1 = f'0,$$

$$A_2 = \tfrac{1}{2!}f''0,$$

$$A_3 = -\tfrac{1}{6}f'0 + \tfrac{1}{3!}f'''0,$$

$$A_4 = +\tfrac{1}{6}f''0 - \tfrac{1}{4!}f^{\mathrm{iv}}0,$$

$$A_5 = +\tfrac{1}{10}f'0 - \tfrac{1}{3!}f'''0 + \tfrac{1}{5!}f^{\mathrm{v}}0.$$

and so on, the general value being expressed as a recurrence formula.

23. Consider the functions

$$\sin(r\sin^{-1}ax) \text{ and } \cos(r\sin^{-1}ax)$$

which for brevity we write $\sin r\theta$ and $\cos r\theta$ respectively.

Represent by the symbol $r^{)n(}$, the central factorial product

$$(r-n)\ldots(r-1)\,r\,(r+1)\ldots(r+n),$$

and by $r_0^{)2n-1(}$ and $r_e^{)2n(}$ the products, respectively,

$$(r-2n+1)\ldots(r-3)(r-1)\,r\,(r+1)(r+3)\ldots(r+2n-1)\,,$$

$$(r-2n)\ldots(r-4)(r-2)\,r\,(r+2)(r+4)\ldots(r+2n)\,.$$

Make the substitutions as below indicated in the composite, putting $x=0$ after differentiation, computing the derivatives from the formulæ

$$\sin(r\sin^{-1}ax) = \Sigma\, r_0^{)2n-1(}\frac{x^{2n-1}}{(2n-1)!}\,,$$

$$\cos(r\sin^{-1}ax) = \Sigma\, r_e^{)2n(}\frac{x^{2n}}{(2n)!}\,,$$

and assuming

$$r^{)-1(} = r \text{ and } r^{)0(} = 1\,.$$

We write down then for Fx the determinant

$$\begin{vmatrix}
fx & ,\ \sin\theta\,, & \frac{\cos 2\theta}{2} & ,\ \sin 3\theta, & \frac{\cos 4\theta}{4} & ,\ \ldots, & \sin(2n-1)\,\theta\,, & \frac{\cos 2n\theta}{2n} \\
\frac{1}{a}f'0 & ,\ 1\ , & 0 & ,\ 3\ , & 0 & ,\ \ldots, & (2n-1) & ,\ 0 \\
\frac{-f''0}{2a^2} & ,\ 0\ , & 1 & ,\ 0\ , & 2 & ,\ \ldots, & 0 & ,\ \frac{(2n)}{2} \\
\frac{+f'''0}{3_0^{)1(}a^3} & ,\ 0\ , & 0 & ,\ 1\ , & 0 & ,\ \ldots, & \frac{(2n-1)_0^{)1(}}{3_0^{)1(}} & ,\ 0 \\
\frac{-f^{\text{iv}}0}{4_e^{)2(}a^4} & ,\ 0\ , & 0 & ,\ 0\ , & 1 & ,\ \ldots, & 0 & ,\ \frac{(2n)_e^{)2(}}{4_e^{)2(}} \\
\frac{+f^{\text{v}}0}{5_0^{)3(}a^5} & ,\ 0\ , & 0 & ,\ 0\ , & 0 & ,\ \ldots, & \frac{(2n-1)_0^{)3(}}{5_0^{)3(}} & ,\ 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\frac{\pm f^{2n-1}0}{(2n-1)_0^{)2n-3(}\,a^{2n-1}} & ,\ 0\ , & 0 & ,\ 0\ , & 0 & ,\ \ldots, & \frac{(2n-1)_0^{)2n-3(}}{(2n-1)_0^{)2n-3(}} & ,\ 0 \\
\frac{\mp f^{2n}0}{(2n)_e^{)2n-2(}a^{2n}} & ,\ 0\ , & 0 & ,\ 0\ , & 0 & ,\ \ldots, & 0 & ,\ \frac{(2n)^{)2n-2(}}{(2n)^{)2n-2(}}
\end{vmatrix}$$

Whence we derive the formula

$$fx - fy = \sum_{r=1}^{r=n}\Big\{ A_{2r-1}[\sin(\overline{2r-1}\sin^{-1}ax) - \sin(\overline{2r-1}\sin^{-1}ay)]$$

$$- \frac{1}{2r}A_{2r}[\cos(2r\sin^{-1}ax) - \cos(2r\sin^{-1}ay)]\Big\}\,; \qquad (106)$$

wherein

$$A_{2r-1} = \left| \begin{array}{ccccc} \dfrac{f^{2n-1}(0)}{(2n-1)_0^{)2n-3(}a^{2n-1}}, & \dfrac{(2n+1)_0^{)2n-3(}}{(2n-1)_0^{)2n-3(}}, & \dfrac{(2n+3)_0^{)2n-3(}}{(2n-1)_0^{)2n-3(}}, & \dfrac{(2n+5)_0^{)2n-3(}}{(2n-1)_0^{)2n-3(}}, & \cdots \\ \dfrac{f^{2n+1}(0)}{(2n+1)_0^{)2n-1(}a^{2n+1}}, & 1 \quad , & \dfrac{(2n+3)_0^{)2n-1(}}{(2n+1)_0^{)2n-1(}}, & \dfrac{(2n+5)_0^{)2n-1(}}{(2n+1)_0^{)2n-1(}}, & \cdots \\ \dfrac{f^{2n+3}}{(2n+3)^{)2n+1(}a^{2n+3}}, & 0 \quad , & 1 \quad , & \dfrac{(2n+5)_0^{)2n+1(}}{(2n+3)_0^{)2n+1(}}, & \cdots \\ \dfrac{f^{2n+5}(0)}{(2n+5)^{)2n+3(}a^{2n+5}}, & 0 \quad , & 0 \quad , & 1 \quad , & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array} \right| ,$$

and

$$A_{2r} = - \left| \begin{array}{ccccc} \dfrac{f^{2n}(0)}{(2n)_e^{)2n-2(}a^{2n}}, & \dfrac{(2n+2)_e^{)2n-2(}}{(2n)_e^{)2n-2(}}, & \dfrac{(2n+4)_e^{)2n-2(}}{(2n)_e^{)2n-2(}}, & \dfrac{(2n+6)_e^{)2n-2(}}{(2n)_e^{)2n-2(}}, & \cdots \\ \dfrac{f^{2n+2}(0)}{(2n+2)_e^{2n}a^{2n+2}}, & 1 \quad , & \dfrac{(2n+4)_e^{)2n(}}{(2n+2)^{)2n(}}, & \dfrac{(2n+6)_e^{)2n(}}{(2n+2)_e^{)2n(}}, & \cdots \\ \dfrac{f^{2n+4}(0)}{(2n+4)_e^{2n+2}a^{2n+4}}, & 0 \quad , & 1 \quad , & \dfrac{(2n+6)_e^{)2n+2(}}{(2n+4)_e^{)2n+2(}}, & \cdots \\ \dfrac{f^{2n+6}(0)}{(2n+6)_e^{2n+4}a^{2n+6}}, & 0 \quad , & 0 \quad , & 1 \quad , & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array} \right| ,$$

simpler forms of (167), which may be obtained through specifying the arbitraries are obvious.

24. Consider the expansion of fx in terms of the functions

$$\varphi_r(x) = x^r e^{ax} \cos bx \,,$$

whose nth derivative is, when $x = 0$ after differentiation,

$$\varphi_r^{\,n}(0) = n_{-1}^{r(} \, (a^2 + b^2)^{\frac{n-r}{2}} \cos\left[(n - r) \tan^1(b/a)\right]$$

or zero, according as n is greater than or less than r.

For brevity, put $a^2 + b^2 = c^2$, and $\theta = \tan^{-1}(b/a)$. Using $n_{-1}^{r(}$, or more simply $n^{r(}$, since the difference -1 is always the same, to denote the descending factorial product

$$n\,(n-1)\,(n-2) \ldots (n - r + 1)\,.$$

Making the substitutions in the composite and reducing to simplest form by factoring, we have

$$\begin{vmatrix} fx & 1 & \varphi_1(x) & \varphi_2(x) & \varphi_3(x) & \ldots & \varphi_n(x) & \varphi_{n+1}(x) \\ fy & 1 & \varphi_1(y) & \varphi_2(y) & \varphi_3(y) & \ldots & \varphi_n(y) & \varphi_{n+1}(y) \\ f'(0) & 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ \frac{f''(0)}{2!} & 0 & \frac{c\cos}{1!} & 1 & 0 & \ldots & 0 & 0 \\ \frac{f'''(0)}{3!} & 0 & \frac{c^2\cos 2\theta}{2!} & \frac{c\cos\theta}{1!} & 1 & \ldots & 0 & 0 \\ \frac{f^{\text{iv}}(0)}{4!} & 0 & \frac{c^3\cos 3\theta}{3!} & \frac{c^2\cos 2\theta}{2!} & \frac{c\cos\theta}{1!} & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{f^n(0)}{n!} & 0 & \frac{c^{n-1}\cos(n-1)\theta}{(n-1)!} & \frac{c^{n-2}\cos(n-2)\theta}{(n-2)!} & \frac{c^{n-3}\cos(n-3)\theta}{(n-3)!} & \ldots & 1 & 0 \\ \Phi(u) & 0 & 0 & 0 & 0 & \ldots & 0 & 1 \end{vmatrix} = 0. \quad (107)$$

Whence

$$fx - fy = \sum_{r=1}^{r=n} A_r\,(x^r e^{ax}\cos bx - y^r e^{ay}\cos by),$$

wherein

$$A_r = (-1)^{r+1}\begin{vmatrix} f'(0) & 1 & 0 & \ldots & 0 \\ \frac{f''(0)}{2!} & c\cos\theta & 1 & \ldots & 0 \\ \frac{f'''(0)}{3!} & \frac{c^2\cos 2\theta}{2!} & c\cos\theta & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{f^r(0)}{r!} & \frac{c\quad\cos(r-1)\theta}{(r-1)!} & \frac{c^{-}\cos(r-2)\theta}{(r-2)!} & \ldots & c\cos\theta \end{vmatrix}.$$

Writing this out, we have

$$A_r = -c\cos\theta\,.\,A_{r-1} + \frac{1}{2!}c^2\cos 2\theta\,.\,A_{r-2} - \,.\,.$$

$$+ (-1)^{r-1}\frac{1}{(r-1)!}c^{r-1}\cos(r-1)\theta\,.\,f'(0) + \frac{1}{r!}f^r(0),$$

and $A_1 = f'(0)$, from which the other values follow at once.

25. Expand the function fx according to the functions

$$\frac{1}{r!}(\varphi x)^r,$$

where we have the existing relation

$$\varphi x = \frac{x - z}{\psi x}.$$

in which z is independent of φx.

Make the substitutions in the composite, and after differentiation put $x = z$, which causes all terms in the body determinant above the diagonal to vanish. The diagonal term in column headed $(\varphi x)^r/(r\,!)$ will be $(\varphi' z)^r$. Divide each row by this diagonal term, so as to make the diagonal unity throughout. This gives for the term below the diagonal term the value $\frac{1}{2}r\,(r+1)\,\varphi'' z/(\varphi' z)^2$, and so on, descending the column, the terms becoming more complicated as we descend; and while they are easily computed, I have not been able to determine a general law for writing them down. Use the symbol $D_z^p \varphi^r$ to mean that the function $(\varphi x)^r/(r\,!)$ has been differentiated p times with respect to x and in the result x replaced by z. We have

$$\left|\begin{array}{llllllll}
fx, & 1, & \varphi x, & \frac{(\varphi x)^2}{2\,!}, & \frac{(\varphi x)^3}{3\,!}, & \frac{(\varphi x)^4}{4\,!}, \ldots, & \frac{(\varphi x)^n}{n\,!}, & \frac{(\varphi x)^{n+1}}{(n+1)\,!} \\
fz, & 1, & 0, & 0, & 0, & 0, \ldots, & 0, & 0 \\
\frac{f'z}{\varphi' z}, & 0, & 1, & 0, & 0, & 0, \ldots, & 0, & 0 \\
\frac{f''z}{(\varphi' z)^2}, & 0, & \frac{\varphi'' z}{(\varphi' z)^2}, & 1, & 0, & 0, \ldots, & 0, & 0 \\
\frac{f'''z}{(\varphi' z)^3}, & 0, & \frac{\varphi''' z}{(\varphi' z)^3}, & 3\frac{\varphi'' z}{(\varphi' z)^2}, & 1, & 0, \ldots, & 0, & 0 \\
\frac{f^{\text{iv}}z}{(\varphi' z)^4}, & 0, & \frac{\varphi^{\text{iv}} z}{(\varphi' z)^4}, & D_z^{\text{iv}}\varphi^2, & 6\frac{\varphi'' z}{(\varphi' z)^2}, & 1, \ldots & 0, & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\frac{f^n z}{(\varphi' z)^n}, & 0, & \frac{D_z^n \varphi}{(\varphi' z)^n}, & \frac{D_z^n \varphi^2}{(\varphi' z)^n}, & \frac{D_z^n \varphi^3}{(\varphi' z)^n}, & \frac{D_z^n \varphi^4}{(\varphi' z)^n}, \ldots, & 1, & 0 \\
\Phi(u), & 0, & 0, & 0, & 0, & 0, \ldots, & 0, & 1
\end{array}\right| = 0. \qquad (108)$$

Expanding this with respect to the first row, we have

$$fx = fz + \sum_1^n A_r \frac{(\varphi x)^r}{r\,!}, \qquad (109)$$

wherein, since

$$\varphi' z = \frac{1}{\psi z}, \qquad \varphi'' z = -2\,\frac{\psi' z}{(\psi z)^2}, \qquad \varphi''' z = \frac{6\,(\psi' z)^2 - 3\psi z\,\psi'' z}{(\psi z)^3},$$

we have

$$A_1 = \frac{f'z}{\varphi'z} = \psi z \,.\, f'z\,, \qquad A_2 = \frac{f'z\,\varphi''z - f''z\,\varphi'z}{(\varphi'z)^3} = -\frac{d}{dz}\,[(\psi z)^2 f'z]\,,$$

$$A_3 = 3\frac{f'z\,(\varphi''z)^2}{(\varphi'z)^5} + \frac{f'''z}{(\varphi'z)^3} - \frac{f'z\,\varphi'''z}{(\varphi'z)^4} - 3\,\frac{f''z\,\varphi''z}{(\varphi'z)^4} = \frac{d^2}{dz^2}\,[(\psi z)^3 f'z]\,.$$

So that we have

$$fx = fz + \varphi x\,[\psi z\, f'z] + \frac{(\varphi x)^2}{2\,!}\frac{d}{dz}[(\psi z)^2 f'z] + \frac{(\varphi x)^3}{3\,!}\frac{d^2}{dz^2}[(\psi z)^3 f'z] + \ldots,$$

which is Lagrange's theorem. I have not identified the coefficients of (109) with those of Lagrange's theorem beyond those here given. The theorem (108) seems to be the same form of Lagrange's theorem as that given by Wronski* in 1812, which has been studied by Professor Cayley, in a paper "On Wronski's Theorem," in the Quarterly Journal of Pure and Applied Mathematics, Vol. XII, No. 47, p. 221, 1873, where Professor Cayley takes up the series in a form which seems to be the same as here given, and proceeds to identify the determinant coefficients with those of Lagrange's Theorem.

To illustrate the method of this last expansion, let us effect the development of $fx = \log x^h$ in terms of the functions

$$\frac{1}{r\,!}\,y^r = \frac{1}{r\,!}\,(x^{h(1-a)} - x^{-ah})^r.$$

Make the substitutions in the composite and after differentiation put $x = 1$, we have

$$\begin{vmatrix} \log x^h, & y, & \dfrac{y^2}{2\,!}, & \dfrac{y^3}{3\,!}, & \ldots, & \dfrac{y^n}{n\,!}, & \dfrac{y^{n+1}}{(n+1)\,!} \\ 1, & 1, & 0, & 0, & \ldots, & 0, & 0 \\ -1, & -2a, & 1, & 0, & \ldots, & 0; & 0 \\ +2\,!, & +3a(a+1), & -6a, & 1, & \ldots, & 0, & 0 \\ -3\,!, & -4a(a+1)(a+2), & 12a(2a+1), & -12a, & \ldots, & 0, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ (-1)^n n\,!, & D_1^n(y), & D_1^n\dfrac{y^2}{2\,!}, & D_1^n\dfrac{y^3}{3\,!}, & \ldots, & 1, & 0 \\ \Phi(u), & 0, & 0, & 0, & \ldots, & 0, & 1 \end{vmatrix} = 0\,, \quad (110)$$

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* At the time this paper was written I was wholly unaware of the work which Wronski has done in the direction of the general expansion of series. Following up Professor Cayley's paper I have made it a point to study Wronski's work on series and find it so interesting that I shall devote a future portion of my paper to an examination and discussion of his work and methods of developing a function in series. See also Emile West's Exposé of Wronski's methods (Gauthier-Villars, Paris, 1886), where this theorem is treated at length.

which, when expanded, gives Dr. McClintock's very general logarithmic series (Am. Jour. Math. Vol. II, No. 2, p. 108).

$$\log x^h = y + (2a - 1)^{1(} \frac{y^2}{2\,!} + (3a - 1)^{2(} \frac{y^3}{3\,!} + (4a - 1)^{3(} \frac{y^4}{4\,!} + \ldots + R.$$

I have deduced this form for the coefficients as far as here written only, and have not attempted to identify the coefficient of y^r in the expansion of (110) with

$$\frac{1}{r\,!}\,(ra - 1)^{r-1(},$$

which I imagine would prove to be a matter of considerable difficulty.

26. We close these illustrations with the determination of the form which an expansion of fx in terms of integral powers of sines or cosines of the variable must have if the development be possible.

Thus, substituting in the composite the functions

$$\frac{1}{r\,!}\sin^r x\,,$$

and using $D_0{}^p s^r$ to indicate the pth derivative of $\frac{1}{r\,!}\sin^r x$ in which after differentiation x is put equal to zero, we have

$$\begin{vmatrix} fx\;, & 1\,, & \sin x, & \frac{\sin^2 x}{2\,!}, & \frac{\sin^3 x}{3\,!}, & \frac{\sin^4 x}{4\,!}, & \ldots, & \frac{\sin^{n+1} x}{(n+1)\,!} \\ f0\;, & 1\,, & 0, & 0\,, & 0\,, & 0\,, & \ldots, & 0 \\ f'0\,, & 0\,, & 1, & 0\,, & 0\,, & 0\,, & \ldots, & 0 \\ f''0\,, & 0\,, & 0, & 1\,, & 0\,, & 0\;. & \ldots, & 0 \\ f'''0, & 0\,, & -1, & 0\,, & 1\,, & 0\,, & \ldots, & 0 \\ f^{\text{iv}}0, & 0\,, & 0, & -4\,, & 0\,, & 1\,, & \ldots, & 0 \\ . & . & . & . & . & . & . & . \\ f^n 0\,, & 0\,, & D_0{}^n s, & D_0{}^n s^2, & D_0{}^n s^3, & D_0{}^n s^4, & \ldots, & 0 \\ \Phi(u), & 0\,, & 0, & 0\,, & 0\,, & 0\,, & \ldots, & 1 \end{vmatrix} = 0 \qquad (111)$$

The expansion of this gives

$$fx = f0 + \sum_1^n A_r \frac{1}{r\,!}\sin^r x + R, \qquad (112)$$

wherein the first five values of A_r are

$$A_1 = 1\,, \quad A_2 = 1, \qquad A_3 = f'0 + f'''0\,,$$

$$A_4 = 4f'' + f^{\text{iv}}0\,, \qquad A_5 = 9f'0 + 10\,f'''0 + f^{\text{v}}0\,,$$

and any number of them are easily computed.

Instead of $\sin^r x$ we may generalize by writing $(\sin 2\pi x/l)^r$, and after differentiation put $x = l$; or we may substitute the functions $(\cos \pi x/l)^r$, and after differentiation put $x = \frac{1}{2}\,l$.

We may also effect the expansion in terms of the functions

$$\frac{1}{(2r-1)!}\sin^{2r-1}x, \quad \frac{1}{(2r)!}\sin^{2r}x\,, \quad \frac{1}{(2r-1)!}\cos^{2r-1}x, \quad \frac{1}{(2r)!}\cos^{2r}x\,,$$

remembering that the first of these is an odd function, while the others are even. Therefore the composite of the first will contain no even derivative rows, and that of the others no odd derivative rows, in virtue of § 4.

For example, using the odd powered sines, we get

$$fx = f0 + \sin x f'0 + \frac{\sin^3 x}{3!}(f'0 + f'''0) + \frac{\sin^5 x}{5!}(9f'0 + 10\,f'''0 + f^{\text{v}}0) + \ldots;$$

and applying this, for sake of illustration, to expand $\sin ax$, we have

$$\sin ax = a \sin x - \frac{a(a^2-1)}{3!}\sin^3 x + \frac{a(a^2-1)(a^2-3^2)}{5!}\sin^5 x + \ldots. \tag{113}$$

If we use the even powered sines we get

$$fx = f0 + \frac{\sin^2 x}{2!}f''0 + \frac{\sin^4 x}{4!}(4\,f''0 + f^{\text{iv}}0) + \ldots.$$

Similarly, the expansion in cosines follows easily. These trigonometrical expansions seem of small value when compared with those of the sines and cosines of multiple angles which occur in the next section.

27. We repeat, finally, that all of the expressions in the form of series herein given are to be considered as having a finite number of terms and a terminal term R as set forth in § 3. They are therefore true equations and functions of n, the number of terms involved. These formulæ are not to be imagined as extending to infinity until it has been demonstrated that R becomes evanescent when n is infinite and the member on the right converges to fx on the left as a limit when n becomes infinitely large, and for what values of the variable the result may be relied upon as arithmetically intelligible and true. That is to say we have definitely determined the true analytical forms, whose

arithmetical values remain yet to be tested. We may thus regard the *qualitative* analysis as having been effected for these formulæ regarded as series; their *quantitative* analysis remains yet to be performed. The terminal term R is a function of an absolutely unknowable value of x, which can only be eliminated by showing that R vanishes when n is infinite. The examples given seem sufficient to illustrate the class of functions to which the section is applied, and we leave its forms for the present in order to pass now to the consideration of those more important functions which yield a difference-product as a body-determinant.

TO BE CONTINUED.

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SOLUTIONS OF EXERCISES.

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DISCUSS the case of two forces acting together upon a particle, one force being directed to the centre, and the other directed to one focus of an ellipse, under the influence of which forces the particle freely describes the ellipse.

[*Yale Prize Problem.*]

SOLUTION.

1. Prof. Curtis, in the Messenger of Mathematics, 1880, proved that, if a particle describe a curve freely under the combined action of the forces F, F', etc., acting along r, r', etc. toward fixed centres, the equation

$$\gamma\varphi d\left[\frac{F}{\varphi}\right] + \gamma'\varphi' d\left[\frac{F'}{\varphi'}\right] + \ldots = 0\,, \quad \text{or} \quad \Sigma\gamma\varphi d\left[\frac{F}{\varphi}\right] = 0$$

must be satisfied at every point of the curve, φ, φ', etc. denoting the forces respectively co-directional with F, F', etc., under which *singly* the given curve would be described; and γ, γ', etc. denoting the corresponding semichords of the circle of curvature at the point.

The proof is as follows: Putting the origin at one of the centres of force, the others being (h, k), (l, m), etc., we have

$$\frac{d^2x}{dt^2} = -F\frac{x}{r} - F'\frac{x-h}{r'} - \ldots, \quad \frac{d^2y}{dt^2} = -F\frac{y}{r} - F'\frac{y-k}{r'} - \ldots.$$